Chapter 4

Farey Series And Related Topics

" ... Beauty is the first test: there is no permanent place in the world for ugly mathematics."

—G.H. Hardy [29], p.85

4.1 Introduction

The *Farey series of order* N, denoted F_N, is the ordered set of all irreducible rational numbers h/k in the interval [0,1] with denominator $k \leq N$. As examples, the Farey series of orders 1 through 7, F_1 through F_7, are shown in (4.1) through (4.7).

$$F_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\} \tag{4.1}$$

$$F_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\} \tag{4.2}$$

$$F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\} \tag{4.3}$$

$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\} \tag{4.4}$$

$$F_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\} \tag{4.5}$$

$$F_6 = \left\{ \frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1} \right\} \tag{4.6}$$

$$F_7 = \left\{ \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1} \right\} \tag{4.7}$$

The distribution of Farey rational numbers in [0,1] is repeated in any $[i, i+1]$, $i \in \mathbb{Z}$; so that the distribution of Farey rationals in [0,1] supplies complete information about the distribution in all of $\mathbb{R}$. We occasionally abuse the proper nomenclature by referring to sequential rational numbers outside the interval [0,1] as Farey terms or as part of F_N, which, technically, they are not. All of the results presented in this chapter, with the exception of results concerning the number of terms, can be shown to apply everywhere in $\mathbb{R}^+$, so this abuse is not harmful.

The study of the Farey series is a topic from number theory (a branch of mathematics). The Farey series finds application in microcontroller work because very often it is economical to linearly scale an integer x using a rational approximation of the form $\lfloor hx/k \rfloor$, $h \in \mathbb{Z}^+$, $k \in \mathbb{N}$ (a single integer multiplication followed by a single integer division with the remainder discarded). The economy of this type of linear scaling comes about because many microcontrollers have integer multiplication and division instructions. However, the technique requires that we be able to choose h and k so as to place h/k as close as possible to the real number r_I that we wish to approximate; always subject to the constraints $h \leq h_{MAX}$ and $k \leq k_{MAX}$ (since microcontroller multiplication and division instructions are constrained in the size of the operands they can accomodate).

Without the relevant results from number theory, it is very difficult to find the rational numbers h/k: $h \in \mathbb{Z}^+, \leq h_{MAX}$, $k \in \mathbb{N}, \leq k_{MAX}$ closest to an arbitrary $r_I \in \mathbb{R}^+$, even for moderate choices of h_{MAX}, k_{MAX} (see Example 4.1). A poorly-written brute-force algorithm might iterate over all h and all k to find the rational numbers closest to r_I; and thus might be approximately $O(h_{MAX} k_{MAX})$. A refined brute-force algorithm might refine the search and be approximately $O(min(h_{MAX}, k_{MAX}))$. However, for implementation on powerful computers which have machine instructions to multiply and divide large integers, or for extended-precision integer arithmetic, even algorithms which are $O(min(h_{MAX}, k_{MAX}))$ are not practical. The best algorithm presented in this work (utilizing the framework of continued fractions), is $O(log\ max(h_{MAX}, k_{MAX}))$, and so is practical even for very large h_{MAX} and k_{MAX}.

Example 4.1: Find the rational numbers h/k which enclose $1/e \approx 0.3678794412$

subject to the constraint $k \leq 2^{16} - 1 = 65\,535$.[1]
Solution: $1/e$ is irrational, and so has left and right neighbors in the Farey series of any order. Using algorithms that will be presented later in the work, these two enclosing rational numbers subject to the constraint $k \leq 2^{16} - 1$ are $18\,089/49\,171$ and $9\,545/25\,946$.

4.2 History Of The Farey Series

The Farey series owes its name to John Farey, a British geologist who in 1816 published the statement to the effect that in the Farey series the middle of any three consecutive terms is the mediant of the other two [69][2] (Thm. 4.10 in this work). However, many mathematicians believe that credit for the Farey series is misplaced, and that the series should not have been named after Farey. In *A Mathematician's Apology* ([29] pp. 81-82), Hardy cites the Farey series as one of the rare examples in scientific history where credit is misplaced:

> " ... Farey is immortal because he failed to understand a theorem which Haros had proved perfectly fourteen years before ... but on the whole the history of science is fair, and this is particularly true in mathematics ... and the men who are remembered are almost always the men who merit it."

Hardy and Wright also provide a footnote about the history of the Farey series, [58], pp. 36-37:

> "The history of the Farey series is very curious. Theorems 28 and 29[3] seem to have been stated and proved first by Haros in 1802; see Dickson, *History*, i. 156. Farey did not publish anything on the subject until 1816, when he stated Theorem 29 in a note in the *Philosophical Magazine*. He gave no proof, and it is unlikely that he had found one, since he seems to have been at the best an indifferent mathematician.
>
> Cauchy, however, saw Farey's statement, and supplied the proof (*Exercices de mathématiques*, i. 114-116). Mathematicians generally have followed Cauchy's example in attributing the results to Farey, and the results will no doubt continue to bear his name."

[69][2] contains the best account of the history of the Farey series on the Web (and contains information on many other interesting topics related to mathematics and number theory, as well). At this site, Dr. Alexander Bogomolny has included John Farey's original letter to the *Philosophical Magazine*, and much

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[1] This example is intended to demonstrate the difficulty of finding suitable rational numbers near an arbitrary r_I without the relevant results from number theory.

[2] Exact URL: `http://www.cut-the-knot.com/blue/FareyHistory.html`.

[3] Theorems 4.8 and 4.10, respectively, in this chapter.

historical and human perspective. This site is highly recommended for anyone who has an interest in mathematics, number theory, and history.

4.3 Properties Of Terms Of The Farey Series

In Section 4.1, we hinted that the properties of the Farey series (which, technically, consists of irreducible rational numbers h/k only in $[0, 1]$) hold in any interval $[n, n+1]$, $n \in \mathbb{N}$. We would first like to show that if $h/k \in [0, 1]$ is irreducible, then any of its corresponding rational numbers $(h + ik)/k$ in $[i, i+1]$, $i \in \mathbb{N}$ are also irreducible.

Theorem 4.2:   Iff h/k, $k \in \mathbb{N}$, $h \in \{0, 1, \dots, k\}$ is irreducible, then

$$\frac{h}{k} + i = \frac{h + ik}{k} \tag{4.8}$$

is also irreducible for $i \in \mathbb{Z}^+$.

Proof: Let $\{p_k^{a_k}\} = p_1^{a_1} p_2^{a_2} \dots p_M^{a_M}$ be the prime factorization of h. Let $\{q_k^{b_k}\} = q_1^{b_1} q_2^{b_2} \dots q_M^{b_M}$ be the prime factorization of k. The coprimality of h and k ensures that $\{p_k^{a_k}\} \bigcap \{q_k^{b_k}\} = \oslash$. We are interested in the irreducibility (or lack thereof) of

$$\frac{h}{k} + i = \frac{p_1^{a_1} p_2^{a_2} \dots p_M^{a_M} + i(q_1^{b_1} q_2^{b_2} \dots q_N^{b_N})}{q_1^{b_1} q_2^{b_2} \dots q_N^{b_N}} \tag{4.9}$$

In order for the expression in (4.9) to be reducible, it is necessary for at least one $q_k \in \{q_1, q_2, \dots, q_N\}$ to divide the numerator, which is possible only if $\{p_k\} \bigcap \{q_k\} \neq \oslash$. (The degenerate cases of $h = 1$ or $k = 1$ are left for the reader as Exercise 4.1.) ■

Remarks: On the other hand, if h/k is reducible, let $\{p_k^{a_k}\} = p_1^{a_1} p_2^{a_2} \dots p_M^{a_M}$ be the prime factors of h which do not appear in k, let $\{q_k^{b_k}\} = q_1^{b_1} q_2^{b_2} \dots q_N^{b_N}$ be the prime factors of k which do not appear in h, and let $\{s_k^{c_k}\} = s_1^{c_1} s_2^{c_2} \dots s_L^{c_L}$ be the prime factors which appear in both h and k. We are interested in the irreducibility (or lack thereof) of

$$\frac{h}{k} + i = \frac{\{s_k^{c_k}\}\{p_k^{a_k}\} + i\{s_k^{c_k}\}\{q_k^{b_k}\}}{\{s_k^{c_k}\}\{q_k^{b_k}\}}. \tag{4.10}$$

It is clear that any choice of $i \in \mathbb{Z}^+$ will allow $\{s_k^{c_k}\}$ to divide both the numerator and the denominator.

Very frequently, it is necessary to compare rational numbers to determine if they are equal; and if not, which is larger. This need occurs both in symbolic manipulation (derivations and proofs), and in calculations. We present a property which is useful for comparison of non-negative rational numbers.

Lemma 4.3: For $a, c \geq 0$ and $b, d > 0$,

$$\frac{a}{b} < \frac{c}{d}, \quad iff \ ad < bc$$

$$\frac{a}{b} = \frac{c}{d}, \quad iff \ ad = bc \tag{4.11}$$

$$\frac{a}{b} > \frac{c}{d}, \quad iff \ ad > bc$$

Proof: Assume $a, c \geq 0$ and $b, d > 0$. Under these assumptions, $a/b < c/d \leftrightarrow a < bc/d \leftrightarrow ad < bc$. Similarly, under the same assumptions, $a/b = c/d \leftrightarrow a = bc/d \leftrightarrow ad = bc$ and $a/b > c/d \leftrightarrow a > bc/d \leftrightarrow ad > bc$. ∎

Remarks: Note it is not required that $a, b, c, d \in \mathbb{Z}$, although this is the way in which the lemma is used exclusively in this work. Note also that the lemma does not cover the case when any of the components a, b, c, d are < 0. For comparing rational numbers with non-negative components, this lemma presents the most convenient method.

Some properties and results concerning the Farey series rely on the *mediant* of two rational numbers, which is defined now.

Definition 4.4 (Mediant): The *mediant* of two [irreducible] rational numbers h/k and h'/k' is the [reduced form of the] fraction

$$\frac{h + h'}{k + k'}. \tag{4.12}$$

Remarks: Note that the mediant of two rational numbers—even irreducible rational numbers—is not necessarily irreducible. For example, the mediant of $1/3$ and $2/3$ is $3/6$, which is not irreducible. Note also that the mediant of two rational numbers is ambiguously defined if we don't require that the rational numbers be irreducible. For example, the mediant of $1/3$ and $2/3$ is $3/6 = 1/2$, but the mediant of $2/6$ and $2/3$ is $4/9$ (a different number than $1/2$). Normally, in this work, we will calculate the mediant only of irreducible rational numbers, and we will define the result to be the reduced form of the fraction calculated.

The mediant of two rational numbers always lies between them in value (but is not necessarily the midpoint). A somewhat stronger statement about mediants can be made, and is presented as Lemma 4.5, below.

Lemma 4.5: For $a, c \geq 0$, $b, d, i, j > 0$, and $a/b < c/d$,

$$\frac{a}{b} < \frac{ia + jc}{ib + jd} < \frac{c}{d}, \tag{4.13}$$

or, equivalently,

$$\frac{ia + jc}{ib + jd} \in \left(\frac{a}{b}, \frac{c}{d}\right). \tag{4.14}$$

Proof: Under the restrictions on $a, b, c, d, i,$ and j; $ad < bc \rightarrow ijad < ijbc \rightarrow i^2ab + ijad < i^2ab + ijbc \rightarrow ia(ib + jd) < ib(ia + jc) \rightarrow ia/ib < (ia + jc)/(ib + jd)$ (employing Lemma 4.3). A similar implication can be used to show that $(ia + jc)/(ib + jd) < jc/jd$. ∎

Remarks: A special case of this lemma is the result that the mediant of two rational numbers is always between them ($i = j = 1$). This lemma gives some insight into the arrangement of intermediate fractions between two convergents (see Chapter 5, *Continued Fractions And Related Topics*).

Theorem 4.6: If h/k and h'/k' are two successive terms of F_N, then $k + k' > N$.

Proof: By Lemma 4.5, the mediant of h/k and h'/k' lies between them, i.e.

$$\frac{h + h'}{k + k'} \in \left(\frac{h}{k}, \frac{h'}{k'}\right). \tag{4.15}$$

Note that if $k + k' \leq N$, the denominator of the mediant, $k + k'$, is less than N, so that either the fraction specified by (4.15) or its reduced form is in F_N; hence there is another term between h/k and h'/k' in F_N. (See [58], Thm. 30, p. 23.) ∎

Theorem 4.7: For $N > 1$, no two consecutive terms of F_N have the same denominator.

Proof: Assume that h/k and h'/k are the two consecutive terms with the same denominator. Note that the only choice of h' which could be the numerator of a consecutive term is $h' = h+1$; otherwise we would have $h/k < (h+1)/k < h'/k$, which implies that h/k and h'/k' are not consecutive, a contradiction. With $h' = h+1$ (the only possibility), let's examine the inequality $h/k < h/(k-1) < (h+1)/k$. $h/k < h/(k-1)$ is always true for any choice of $k > 1$. It can be shown using Lemma 4.3 that $h/(k-1) < (h+1)/k$ is true iff $h < k-1$. So, for any $h \in \{2, \ldots, k-2\}$, we can always use the fraction $h/(k-1)$ as an intermediate term to show that h/k and $(h+1)/k$ are not consecutive in F_N. If $h = k-1$, then we are considering the two fractions $(k-1)/k$ and k/k, and k/k cannot

be in lowest terms—after reducing k/k the two fractions being considered no longer have the same denominator. (See [58], Thm. 31, p. 24.) ∎

Theorem 4.8: If h/k and h'/k' are two successive terms of F_N, then

$$h'k - hk' = 1. \tag{4.16}$$

Proof: For any h/k, Lemma 3.6 guarantees that an h'/k' satisfying (4.16) exists. If $h' = x_0, k' = y_0$ is a solution, then $h' = x_0 + rh, k' = y_0 + rk, r \in \mathbb{Z}$ is also a solution, and Lemma 3.5 guarantees that any h', k' so chosen will be coprime. Note that r can be chosen so that $0 \leq N - k < k' \leq N$, and that h'/k' will be $\in F_N$.

However, it still needs to be established that h'/k' is the *next* term in F_N (i.e. that there can be no intervening terms). To show this, assume that an intervening term a/b exists, so that $h/k < a/b < h'/k'$, with $b \leq N$. In this case, the distance from h/k to a/b is

$$\frac{a}{b} - \frac{h}{k} = \frac{ak - bh}{bk} \geq \frac{1}{bk}. \tag{4.17}$$

Similarly, the distance from a/b to h'/k' is

$$\frac{h'}{k'} - \frac{a}{b} = \frac{h'b - k'a}{k'b} \geq \frac{1}{k'b}. \tag{4.18}$$

(The inequalities in Eqns. 4.17 and 4.18 come about through Lemma 4.3—the numerator in each case must be at least 1 if it is assumed $h/k < a/b < h'/k'$.)

The distance from h/k to h'/k' is

$$\frac{h'}{k'} - \frac{h}{k} = \frac{h'k - hk'}{kk'} = \frac{1}{kk'}. \tag{4.19}$$

The distance from h/k to h'/k' must be the sum of the distances from h/k to a/b and from a/b to h'/k':

$$\left(\frac{h'}{k'} - \frac{h}{k}\right) = \left(\frac{a}{b} - \frac{h}{k}\right) + \left(\frac{h'}{k'} - \frac{a}{b}\right). \tag{4.20}$$

Substituting (4.17), (4.18), and (4.19) into (4.20) leads to

$$\frac{1}{kk'} \geq \frac{1}{bk} + \frac{1}{k'b} = \frac{k'}{bkk'} + \frac{k}{bkk'} = \frac{k' + k}{bkk'} > \frac{N}{bkk'} > \frac{1}{kk'}, \tag{4.21}$$

a contradiction. Therefore, a/b must be h'/k' and $h'k - hk' = 1$. (See [58], Thm. 28, p. 23, and the second proof on pp. 25-26.) ∎

Lemma 4.9: For h/k, h'/k' and h''/k''; $h, h', h'' \in \mathbb{Z}^+$, $k, k', k'' \in \mathbb{N}$, with

$$h'k - hk' = h''k' - h'k'' = 1, \tag{4.22}$$

$$(k' > k'') \rightarrow \left(\frac{h'}{k'} - \frac{h}{k} < \frac{h''}{k''} - \frac{h}{k} \right). \tag{4.23}$$

Proof:

$$\frac{h'}{k'} - \frac{h}{k} = \frac{1}{kk'} \tag{4.24}$$

$$\frac{h''}{k''} - \frac{h}{k} = \frac{1}{kk''} \tag{4.25}$$

$$(k' > k'') \rightarrow \left(\frac{1}{kk'} < \frac{1}{kk''} \right) \tag{4.26}$$

∎

Remarks: This lemma essentially says that if more than one potential successor to h/k in F_N, h'/k' and h''/k'', both meet the necessary test provided by Theorem 4.8, the potential successor with the largest denominator is the successor, because it is closer to h/k. This lemma is used in proving Theorem 4.11.

Theorem 4.10:   If h/k, h'/k', and h''/k'' are three successive terms of F_N, then

$$\frac{h'}{k'} = \frac{h + h''}{k + k''}. \tag{4.27}$$

Proof: Starting from Theorem 4.8:

$$h'k - hk' = h''k' - h'k'' = 1 \tag{4.28}$$

$$h'(k + k'') = k'(h + h'') \tag{4.29}$$

$$\frac{h'}{k'} = \frac{h + h''}{k + k''}. \tag{4.30}$$

∎

4.4 Generation Of Terms Of The Farey Series

Earlier sections of this chapter have enumerated important properties of the Farey series. However, these properties are of limited practical value because they don't help to solve practical problems in microcontroller work—one would like to be able to generate (in order, of course) all of the terms of a Farey series so that one can choose suitable terms near some $r_I \in \mathbb{R}^+$ that one wishes to approximate with $r_A = h/k$.

Fortunately, the properties presented in earlier sections do allow the generation of successive Farey terms, as the following theorem shows.

Theorem 4.11: For a Farey series of order N,

$$F_N = \left\{ \frac{h_0}{k_0}, \frac{h_1}{k_1}, \frac{h_2}{k_2}, \frac{h_3}{k_3}, \ldots \right\}, \tag{4.31}$$

the recursive relationships in (4.32), (4.33), (4.34), and (4.35) apply.

$$h_j = \left\lfloor \frac{k_{j-2} + N}{k_{j-1}} \right\rfloor h_{j-1} - h_{j-2} \tag{4.32}$$

$$k_j = \left\lfloor \frac{k_{j-2} + N}{k_{j-1}} \right\rfloor k_{j-1} - k_{j-2} \tag{4.33}$$

$$h_j = \left\lfloor \frac{k_{j+2} + N}{k_{j+1}} \right\rfloor h_{j+1} - h_{j+2} \tag{4.34}$$

$$k_j = \left\lfloor \frac{k_{j+2} + N}{k_{j+1}} \right\rfloor k_{j+1} - k_{j+2} \tag{4.35}$$

Proof: Only (4.32) and (4.33) are proved—(4.34) and (4.35) come by symmetry.

Assume that h_{j-2}/k_{j-2} and h_{j-1}/k_{j-1} are known and we wish to find h_j/k_j.

Note that by Theorem 4.8,

$$h_{j-1}k_{j-2} - h_{j-2}k_{j-1} = 1 \tag{4.36}$$

and

$$h_j k_{j-1} - h_{j-1} k_j = 1. \tag{4.37}$$

We desire to identify the set of rational numbers $\{h_j/k_j\}$ that satisfy (4.37). If this set is identified, by Lemma 4.9, we can simply choose the member of the set that has the largest denominator.

Choose as trial solutions

$$h_j = i h_{j-1} - h_{j-2} \tag{4.38}$$

and

$$k_j = i k_{j-1} - k_{j-2}, \tag{4.39}$$

where $i \in \mathbb{Z}$ is an integer parameter; i.e. define as the trial set of solutions all h_j/k_j that can be formed by $i \in \mathbb{Z}$ substituted into (4.38) and (4.39). Substitution of this trial solution into (4.37) yields

$$(ih_{j-1} - h_{j-2})k_{j-1} - h_{j-1}(ik_{j-1} - k_{j-2}) = h_{j-1}k_{j-2} - h_{j-2}k_{j-1} = 1. \tag{4.40}$$

Thus, any solution in the form of (4.38, 4.39) will meet the necessary test posed by Theorem 4.8.

However, we must also demonstrate that solutions of the form suggested by (4.38, 4.39) are the *only* solutions which meet the necessary test posed by Theorem 4.8, and also demonstrate how to pick the solution of this form with the largest denominator $k_j \leq N$.

To demonstrate that (4.38, 4.39) are the only solutions, solve (4.37) for h_j, yielding

$$h_j = \frac{h_{j-1}}{k_{j-1}} k_j + \frac{1}{k_{j-1}}. \tag{4.41}$$

Since h_{j-1} and k_{j-1} are known, it is clear that (4.41) defines a required linear relationship between h_j and k_j, and that the only solutions are the values of h_j and k_j meeting (4.41) which are integers. Assume that a particular integer solution $\overline{h_j}, \overline{k_j}$ to (4.41) is known and that a second integer solution $\widehat{h_j}, \widehat{k_j}$ is sought. In order for $\widehat{h_j}$ to be an integer, it must differ from $\overline{h_j}$ by an integer, implying

$$\frac{h_{j-1}}{k_{j-1}}(\widehat{k_j} - \overline{k_j}) \in \mathbb{Z}. \tag{4.42}$$

It is easy to see from the form of (4.42) that because h_{j-1} and k_{j-1} are coprime, in order for (4.42) to be met, $\widehat{k_j} - \overline{k_j}$ must contain every prime factor of k_{j-1} in at least equal multiplicity, implying $|\widehat{k_j} - \overline{k_j}| \geq k_{j-1}$. It follows that (4.38, 4.39) are the only solutions which meet the necessary test posed by Theorem 4.8. We only need find the solution with the largest denominator.

Solving $ik_{j-1} - k_{j-2} \leq N$ for the largest integral value of i leads to

$$i = \left\lfloor \frac{k_{j-2} + N}{k_{j-1}} \right\rfloor \tag{4.43}$$

and directly to (4.32) and (4.33). ∎

Given two consecutive Farey terms, Theorem 4.11 suggests a way to generate as many neighboring terms as desired in either the descending or ascending direction. Note that at an integer i, $(iN - 1)/N$, $i/1$, and $(iN + 1)/N$ are always consecutive terms in F_N; thus it is typically convenient to build the Farey series starting at an integer. This method is presented as Algorithm 4.12.

Algorithm 4.12 ($O(N^2)$ Exhaustive Construction Algorithm For Finding Enclosing Neighbors To r_I In F_N):

- Choose $i = \lfloor r_I + 1/2 \rfloor$ as the integer from which to construct consecutive Farey terms (i.e. the nearest integer to r_I).

- Choose $i/1$ and $(iN+1)/N$ as the two consecutive Farey terms from which to start construction.

- Use (4.32) and (4.33) or (4.34) and (4.35) to construct Farey terms in the increasing or decreasing direction until r_I is enclosed.

4.5 Number Of Terms In The Farey Series

The number of terms[4] $C(N)$ in the Farey series of order N, F_N, is

$$C(N) = 1 + \sum_{k=1}^{N} \phi(k), \tag{4.44}$$

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[4]In the interval $[0, 1]$ only.

where $\phi(\cdot)$ is Euler's totient function [70]. The asymptotic limit for this function $C(N)$ is

$$C(N) \sim \frac{3N^2}{\pi^2} \approx 0.3040N^2. \tag{4.45}$$

Because the number of terms in F_N is approximately $O(N^2)$, the method presented in the previous section (Algorithm 4.12) is only practical for moderate N (a few hundred or less). For Farey series of large order, building the Farey series until r_I is enclosed is not a practical method.

For example, $F_{2^{32}-1}$ (a Farey series of interest for computer work, because many processors can multiply 32-bit integers) contains about 1.8×10^{19} elements. Even using a computer that could generate 10^9 Farey terms per second (an unrealistic assumption at the time of this writing), building $F_{2^{32}-1}$ between two consecutive integers would require over 500 years. It is also noteworthy that certainly $2^{32} - 1$ is not an upper bound on the order of Farey series that are useful in practice—with scientific computers, it may be advantageous to be able to find best approximations in $F_{2^{64}-1}$, $F_{2^{128}-1}$, or Farey series of even higher order.

Algorithm 4.12 is useful to illustrate concepts, or to find best approximations in Farey series of moderate order. However, for Farey series of large order, this algorithm isn't usable. Chapter 5, *Continued Fractions And Related Topics*, presents algorithms based on the framework of continued fractions which are suitable for finding best rational approximations in Farey series of large order.

4.6   Probabilistic Results Surrounding $|r_I - r_A|$

If rational numbers of the form $r_A = h/k$, subject to the constraint $k \leq k_{MAX}$, are used to approximate arbitrary real numbers r_I, it might not be clear how close we can "typically" choose r_A to an aribtrary r_I under the constraint. We consider different asymptotics for the precision of the approximation of an arbitrary r_I by a rational number $r_A = h/k$ with $k \leq k_{MAX}$. For simplicity of notation we denote $\alpha = r_I$ and $N = k_{\text{MAX}}$ and assume, without loss of generality, that $\alpha \in [0, 1]$.

We are thus interested in the asymptotic behaviour, when $N \to \infty$, of the quantity

$$\rho_N(\alpha) = \min_{h/k \in F_N} |\alpha - h/k| \,,$$

which is the distance between α and F_N, the Farey series of order N.

The worst–case scenario is not very interesting: from the construction of the Farey series we observe that for a fixed N the longest intervals between the neighbours of F_N are $[0, 1/N]$ and $[1 - 1/N, 1]$ and therefore for all N,

$$\max_{\alpha \in [0,1]} \rho_N(\alpha) = \frac{1}{2N} \,. \tag{4.46}$$

(This supremum is achieved at the points $1/(2N)$ and $1 - 1/(2N)$.)

However, such behaviour of $\rho_N(\alpha)$ is not typical: as is shown below, typical values of the approximation error $\rho_N(\alpha)$ are much smaller.

First consider the behaviour of $\rho_N(\alpha)$ for almost all $\alpha \in [0,1]$.[5] We then have (see [64], [65]) that for almost all $\alpha \in [0,1]$ and any $\varepsilon > 0$, (4.47) and (4.48) hold.

$$\lim_{N\to\infty} \rho_N(\alpha)N^2 \log^{1+\varepsilon} N = +\infty, \quad \liminf_{N\to\infty} \rho_N(\alpha)N^2 \log N = 0 \qquad (4.47)$$

$$\limsup_{N\to\infty} \frac{\rho_N(\alpha)N^2}{\log N} = +\infty, \quad \lim_{N\to\infty} \frac{\rho_N(\alpha)N^2}{\log^{1+\varepsilon} N} = 0 \qquad (4.48)$$

Even more is true: in (4.47) and (4.48) one can replace $\log N$ by $\log N \log \log N$, $\log N \log \log N \log \log \log N$, and so on. Analogously, $\log^{1+\varepsilon} N$ could be replaced by $\log N (\log \log N)^{1+\varepsilon}$, $\log N \log \log N (\log \log \log N)^{1+\varepsilon}$, and so on.

These statements are analogues of Khinchin's metric theorem, the classic result in metric number theory, see e.g. [64].

The asymptotic distribution of the suitably normalized $\rho_N(\alpha)$ was derived in [66]. A main result of this paper is that the sequence of functions $N^2 \rho_N(\alpha)$ converges in distribution, when $N \to \infty$, to the probability measure on $[0, \infty)$ with the density given by (4.49).

$$p(\tau) = \begin{cases} 6/\pi^2, & \text{if } 0 \leq \tau \leq \frac{1}{2} \\[2mm] \frac{6}{\pi^2 \tau}\left(1 + \log \tau - \tau\right), & \text{if } \frac{1}{2} \leq \tau \leq 2 \\[2mm] \frac{3}{\pi^2 \tau}\left(2\log(2\tau) - 4\log(\sqrt{\tau} + \sqrt{\tau - 2}) - (\sqrt{\tau} - \sqrt{\tau - 2})^2\right), & \\ & \text{if } 2 \leq \tau < \infty \end{cases} \qquad (4.49)$$

This means that for all a, A such that $0 < a < A < \infty$, (4.50) applies, where 'meas' denotes for the standard Lebesgue measure on $[0, 1]$.

$$\text{meas}\{\alpha \in [0,1] : a < N^2 \rho_N(\alpha) \leq A\} \to \int_a^A p(\tau)d\tau \quad \text{as } N \to \infty \qquad (4.50)$$

Another result in [66] concerns the asymptotic behavior of the moments of the approximation error $\rho_N(\alpha)$. It says that for any $\delta \neq 0$ and $N \to \infty$, (4.51) applies, where $\zeta(\cdot)$ and $B(\cdot, \cdot)$ are the Riemann zeta–function and the Beta–function, respectively.

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[5]A statement is true for almost all $\alpha \in [0,1]$ if the measure of the set where this statement is wrong has measure zero.

$$\frac{\delta+1}{2}\int_0^1 \rho_N^\delta(\alpha)d\alpha = \begin{cases} \infty, & \text{if } \delta \leq -1 \\[2ex] \frac{3}{\delta^2\pi^2}\left(2^{-\delta}+\delta 2^{\delta+2}\mathrm{B}(-\delta,\tfrac{1}{2})\right)N^{-2\delta}\left(1+o(1)\right), \\ & \text{if } -1<\delta<1, \delta\neq 0 \\[2ex] \frac{3}{\pi^2}N^{-2}\log N + O\left(N^{-2}\right), & \text{if } \delta = 1 \\[2ex] 2^{-\delta}\frac{\zeta(\delta)}{\zeta(\delta+1)}N^{-\delta-1}+O\left(N^{-2\delta}\right), & \text{if } \delta > 1 \end{cases} \qquad (4.51)$$

In particular, the average of the approximation error $\rho_N(\alpha)$ asymptotically equals

$$\int_0^1 \rho_N(\alpha)d\alpha = \frac{3}{\pi^2}\frac{\log N}{N^2} + O\left(\frac{1}{N^2}\right), \quad N\to\infty. \qquad (4.52)$$

Comparison of (4.52) with (4.48) shows that the asymptotic behavior of the average approximation error $\int \rho_N(\alpha)d\alpha$ resembles the behavior of the superior limit of $\rho_N(\alpha)$. Even this limit decreases much faster than the maximum error $\max_\alpha \rho_N(\alpha)$, see (4.46): for typical α the rate of decrease of $\rho_N(\alpha)$, when $N\to\infty$, is, roughly speaking, $1/N^2$ rather than $1/N$, the error for the worst combination of α and N.

These results show that there is a significant advantage to using the Farey series as the set from which to choose rational approximations, rather than more naively using only rational numbers with the maximum denominator k_{MAX} (as is often done in practice).

4.7   Integer Lattice Interpretation Of The Farey Series

The Farey series has a convenient and intuitive graphical interpretation involving the integer lattice (see Fig. 4.1, which illustrates this interpretation, but with h also restricted). (By integer lattice, we mean the $\mathbb{R}^2$ plane with each point (x,y), $x,y\in\mathbb{Z}$, marked.) In such an interpretation, each rational number h/k corresponds to a point (k,h) which is h units above and k units to the right of the origin.

From the graphical interpretation suggested by Fig. 4.1, the following properties are intuitively clear.

- The angle of a ray drawn from the origin to the point (k,h) corresponding to the rational number h/k is $\theta = tan^{-1}\, h/k$.

- Any integer lattice point on a line from the origin drawn at the angle θ has the value $h/k = tan\,\theta$. All points corresponding to rational numbers with the same value will be on such a line, and thus form an equivalence class.
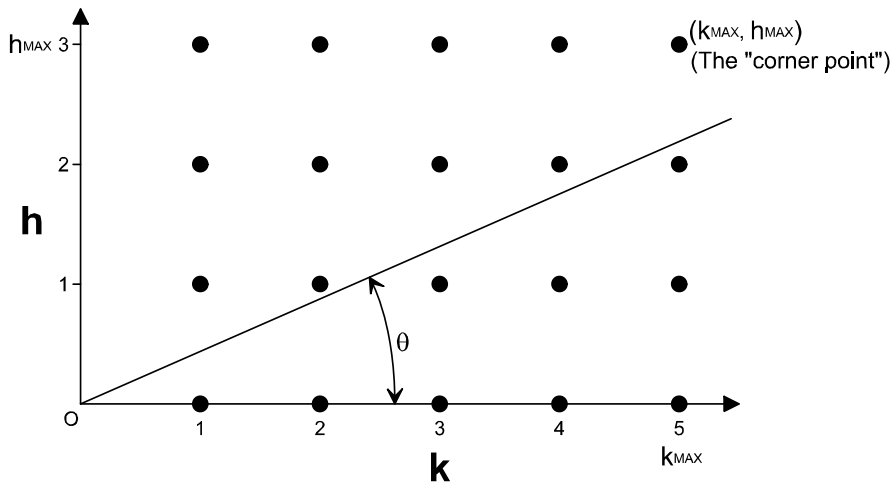
Figure 4.1: Graphical Interpretation Of Rational Numbers h/k That Can Be Formed With $h \leq h_{MAX} = 3$, $k \leq k_{MAX} = 5$

- A rational number h/k is irreducible iff its corresponding point (k, h) is "directly" visible from the origin with no intervening points.

- The Farey series of order N, F_N, can be formed graphically by starting with the set of integer lattice points (k, h) : $h \in \mathbb{Z}^+ \wedge 1 \leq k \leq N$, then sweeping a line extended from the origin, starting with angle $\theta = 0$, through $0 \leq \theta < \pi/2$, and recording in order each point directly visible from the origin.[6]

Fig. 4.2 illustrates the graphical construction method of F_5. Note that only integer lattice points which are directly visible from the origin (with no intervening points) are selected. (Fig. 4.2, like Fig. 4.1, shows the case of constrained h—the integer lattice should be continued "upward" to construct the Farey series.)

4.8 Generating $F_{k_{MAX},\overline{h_{MAX}}}$ Over A Rectangular Region Of The Integer Lattice

In practice, F_N does not represent the set of rational numbers that may be used for rational approximation in an application; hence it isn't usually appropriate to choose a rational number from F_N strictly as the theory of numbers defines it. An actual application is parameterized not just by k_{MAX} (the maximum

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[6]Note that Fig. 4.1, because it illustrates the case when h is constrained as well, does not show integer lattice points for $h > h_{MAX}$. In principle, if the integer lattice shown in Fig. 4.1 were extended indefinitely "upward", every positive irreducible rational number with $k \leq k_{MAX} = 5$ could be found graphically.
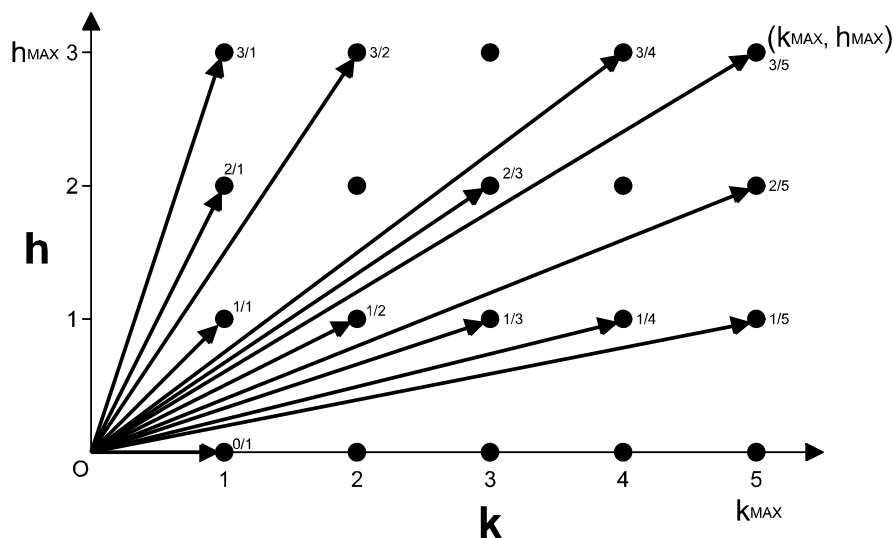
Figure 4.2: Graphical Interpretation Of Irreducible Rational Numbers h/k That Can Be Formed With $h \leq h_{MAX} = 3$, $k \leq k_{MAX} = 5$

denominator that may be chosen, which is considered in the definition of the Farey series), but also by h_{MAX} (the maximum numerator that may be chosen). Typically, h_{MAX} exists as a constraint because a machine multiplication instruction is limited in the size of the operands it can accomodate; and k_{MAX} exists as a constraint because a machine division instruction is limited in the size of the divisor it can accomodate.

In practice, the rational numbers that may be used for rational approximation represent a rectangular region of the integer lattice—all $(k, h) : h \leq h_{MAX} \wedge k \leq k_{MAX}$ (Figs. 4.1, 4.2).

Fig. 4.1 supplies a graphical interpretation of all rational numbers that can be formed with constraints $h \leq h_{MAX} = 3$ and $k \leq k_{MAX} = 5$. Each point of the integer lattice shown in the figure is a rational number, not necessarily irreducible. Because under this graphical interpretation a rational number is irreducible iff it can be reached by a ray from the origin with no intervening rational numbers, it is clear that the complete ordered set of irreducible rational numbers that can be formed under the constraints $h \leq h_{MAX}$ and $k \leq k_{MAX}$ can be obtained graphically by sweeping a ray from the origin through the angles $0 \leq \theta < \pi/2$, recording each point directly visible from the origin. This graphical construction process is illustrated in Fig. 4.2.

From the graphical construction process shown in Fig. 4.2, it can be seen that the set of irreducible rational numbers that can be formed subject to the constraints $h \leq h_{MAX} \wedge k \leq k_{MAX}$ is:

$$\left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1} \right\} . \tag{4.53}$$

We denote the ordered set of irreducible rational numbers that can be formed subject to the constraints $h \leq h_{MAX} \wedge k \leq k_{MAX}$ as $F_{k_{MAX}, \overline{h_{MAX}}}$.[7]

There are three important questions to be asked about the series $F_{k_{MAX}, \overline{h_{MAX}}}$:

- What are the smallest and largest rational numbers in $F_{k_{MAX}, \overline{h_{MAX}}}$? (This question is easy: the smallest two rational numbers in $F_{k_{MAX}, \overline{h_{MAX}}}$ are $0/1$ and $1/k_{MAX}$, and the largest rational number is $h_{MAX}/1$.)

- How do we construct $F_{k_{MAX}, \overline{h_{MAX}}}$?

- If we desire to approximate a real number r_I, $r_{IMIN} \leq r_I \leq r_{IMAX}$, using a rational number selected from $F_{k_{MAX}, \overline{h_{MAX}}}$, how large might the error $|h/k - r_I|$ be?

4.8.1   Construction Of $F_{k_{MAX}, \overline{h_{MAX}}}$

To construct $F_{k_{MAX}, \overline{h_{MAX}}}$, for $0 \leq \theta \leq \tan^{-1}(h_{MAX}/k_{MAX})$—the region of the series where k_{MAX} is the dominant constraint, i.e. below the "corner point" in Figs. 4.1 and 4.2—note that these terms are simply $F_{k_{MAX}}$ up to h_{MAX}/k_{MAX} or its reduced equivalent.[8] To construct $F_{k_{MAX}, \overline{h_{MAX}}}$ for $\tan^{-1}(h_{MAX}/k_{MAX}) < \theta < \pi/2$—the region of the series where h_{MAX} is the dominant constraint, i.e. above the "corner point" in Figs. 4.1 and 4.2—note that by a graphical argument of symmetry, these terms are the reciprocals of ascending terms of $F_{h_{MAX}}$. For example, in Fig. 4.2, if the h- and k- axes are transposed, it is easy to see that $3/1$ in the original integer lattice would correspond to $1/3$ in the transposed integer lattice. This argument of symmetry immediately suggests a procedure for constructing $F_{k_{MAX}, \overline{h_{MAX}}}$:

- Construct $F_{k_{MAX}}$ from $0/1$ up through h_{MAX}/k_{MAX} or its reduced equivalent.

- Construct $F_{h_{MAX}}$ from $1/h_{MAX}$ up to k_{MAX}/h_{MAX} or its reduced equivalent; then reverse the order of the terms and take the reciprocal of each term.

- Concatenate the results from the two steps above.

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[7]Notationally, in general, we use an overbar on the order of a Farey series to denote that the terms are inverted and reversed in order. For example, $F_3 = \{0/1, 1/3, 1/2, \dots\}$, but $F_{\overline{3}} = \{\dots, 2/1, 3/1\}$. Notation such as $F_{A, \overline{B}}$ is an extension of that convention.

[8]If this is not intuitively clear, note in Figs. 4.1 and 4.2 that all of the terms of $F_{k_{MAX}}$—that is, all rational numbers, both reducible and irreducible, with $k \leq k_{MAX}$—are available for selection until the "corner point" is reached.

Example 4.13:   Construct $F_{5,\overline{3}}$, the set of all irreducible rational numbers h/k that can be formed with $h \le h_{MAX} = 3$ and $k \le k_{MAX} = 5$.[9]
Solution: (Using the method of construction presented above.) First, F_5 up through $h_{MAX}/k_{MAX} = 3/5$ or its reduced equivalent should be constructed. This series is

$$F_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \dots \right\}. \tag{4.54}$$

Second, F_3 is constructed from $1/3$ up to $k_{MAX}/h_{MAX} = 5/3$ or its reduced equivalent (not including the final term, $5/3$). This series is

$$F_3 = \left\{ \dots, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \dots \right\}. \tag{4.55}$$

Third, the terms of F_3 are reversed in order, and the reciprocal of each term is calculated, yielding

$$F_{\overline{3}} = \left\{ \dots, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1} \right\}. \tag{4.56}$$

Finally, concatenating (4.54) and (4.56) yields $F_{5,\overline{3}}$, below.

$$F_{5,\overline{3}} = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1} \right\} \tag{4.57}$$

It is clear that Thm. 4.11 and (4.32) through (4.35) can be used to construct $F_{k_{MAX},\overline{h_{MAX}}}$. However, such algorithms are not discussed because—even with refinements—they can be no better than $O(N)$ and are not fruitful to develop. Instead, an $O(logN)$ algorithm is presented in Chapter 5, *Continued Fractions And Related Topics*.

4.8.2   Distance Between Terms Of $F_{h_{MAX},k_{MAX}}$

The maximum *distance* between terms of $F_{h_{MAX},k_{MAX}}$ also establishes what we call the maximum *placement error*, $|r_A - r_I|$, in choosing $r_A = h/k$. Specifically, the maximum distance is twice the maximum placement error. Clearly, with a maximum distance specified, choosing $r_I = (x+y)/2$ for two successive terms x

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[9]Note that $F_{5,\overline{3}}$ is the series depicted in Fig. 4.2, and this example can be verified against the figure.
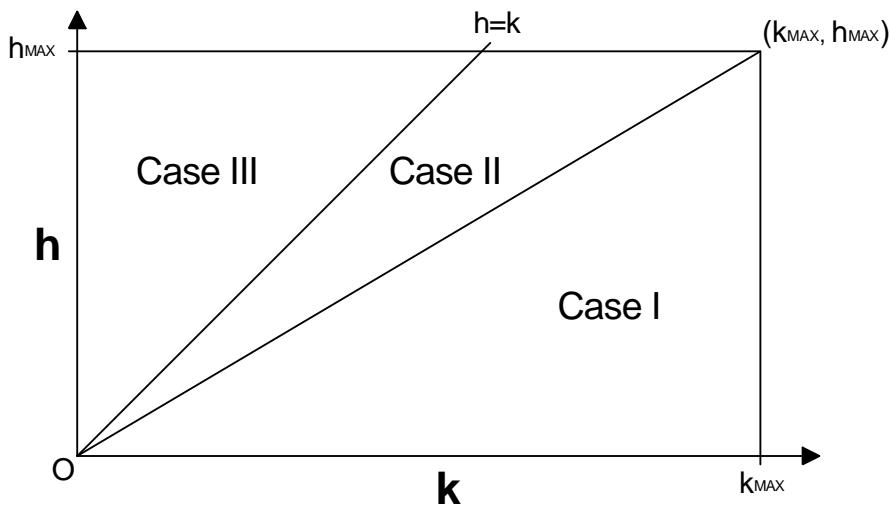
Figure 4.3: Three Cases For Bounding Distance Between Terms In $F_{k_{MAX},\overline{h_{MAX}}}$

and y separated by the maximum distance is the most antagonistic choice of r_I possible. We use the two notions (maximum distance and maximum placement error) interchangeably and don't bother to convert between them, as they are the same notion and differ only by a factor of two.

It is clear from the earlier discussion of the Farey series that the maximum distance between terms in $F_{k_{MAX}}$ is $1/k_{MAX}$, and that this maximum distance occurs only adjacent to an integer. It is also clear from the discussion of $F_{\overline{h_{MAX}}}$ that the maximum distance between terms is 1.

Thus, when we use $F_{k_{MAX},\overline{h_{MAX}}}$ to approximate real numbers, in general the worst-case distance between terms is 1.

In practical applications when rational approximation is used, the approximation tends to be used over a restricted interval $[l \gg 0, r \ll h_{MAX}]$ rather than over the full range of the rational numbers that can be formed, $[0, h_{MAX}]$. This section develops novel upper bounds on the distance between terms of $F_{k_{MAX},\overline{h_{MAX}}}$ in an interval $[l, r]$. For simplicity, assume $l, r \in F_{k_{MAX},\overline{h_{MAX}}}$.

Three distinct cases are developed (Figure 4.3). The upper bound developed from Case III is always larger than the upper bound developed from Case II, which is always larger than the upper bound developed from Case I; so if only the absolute maximum error over the interval $[l, r]$ is of interest, only the highest-numbered case which applies needs to be evaluated. However, some applications may have different error requirements in different regions of the interval $[l, r]$, and for these applications it may be beneficial to analyze more than one case.

Case I: $r_I < h_{MAX}/k_{MAX}$

With $r_I < h_{MAX}/k_{MAX}$, $k \leq k_{MAX}$ is the dominant constraint, and the neighbors available to r_I are simply the terms of $F_{k_{MAX}}$. If $[l,r] \cap [0, h_{MAX}/k_{MAX}]$ includes an integer, clearly the maximum distance from r_I to the nearest available term of $F_{k_{MAX},\overline{h_{MAX}}}$ is given by

$$\left| \frac{h}{k} - r_I \right| \leq \frac{1}{2k_{MAX}}, \tag{4.58}$$

which is the same result for the Farey series in general.

If $[l,r] \cap [0, h_{MAX}/k_{MAX}]$ does not include an integer, it can be shown that the maximum distance between Farey terms is driven by the rational number with the smallest denominator in the interval $[l,r]$.

For two consecutive terms p/q and p'/q' in $F_{k_{MAX}}$, $p'q - pq' = 1$ (Thm. 4.8), so that

$$\frac{p'}{q'} - \frac{p}{q} = \frac{p'q - pq'}{qq'} = \frac{1}{qq'}. \tag{4.59}$$

By Thm. 4.6, $q + q' > k_{MAX}$, therefore

$$\frac{1}{qk_{MAX}} \leq \frac{1}{qq'} < \frac{1}{q(k_{MAX} - q)}. \tag{4.60}$$

Let q_{MIN} be the smallest denominator of any rational number $\in F_{k_{MAX}}$ in the interval $[l,r]$. It is then easy to show that for any consecutive denominators q, q' which occur in $F_{k_{MAX}}$ in the interval $[l,r]$,

$$\frac{1}{qq'} < \frac{1}{q_{MIN}\ max(q_{MIN}, k_{MAX} - q_{MIN})}. \tag{4.61}$$

Thus, the upper bound on the distance between consecutive terms of $F_{k_{MAX}}$ in an interval $[l,r]$ is tied to the minimum denominator of any rational number $\in F_{k_{MAX}}$ in $[l,r]$.

Note that clearly $q_{MIN} \leq 1/(r - l)$, so for most practical intervals $[l,r]$, the search for q_{MIN} would not be computationally expensive. However, applications could arise where an approximation is used in an *extremely* narrow interval, and having an algorithm available that is computationally viable for such cases is advantageous. For example, locating the rational number $\in F_{2^{20,000}}$ with the smallest denominator in an interval of width $2^{-10,000}$ could be a serious computational problem.

To locate q_{MIN} in $[l,r]$, note that at least one rational number with q_{MIN} as a denominator in $[l,r]$ is the best approximation of order q_{MIN} to the midpoint

of the interval, $(l+r)/2$.[10] By theorem ([59], Theorem 15), every best approximation of a number is a convergent or intermediate fraction of the continued fraction representation of the number. We seek the convergent or intermediate fraction of $(l+r)/2$ with the smallest denominator that is in the interval $[l, r]$.[11]

The convergents and intermediate fractions of $(l+r)/2$ are naturally arranged in order of increasing denominator. However, it would be inefficient to test *every* intermediate fraction for membership in $[l, r]$, as partial quotients a_k are unlimited in size and such an algorithm may not be $O(log \ k_{MAX})$. Instead, since intermediate fractions are formed using the parameterized expression $(ip_k + p_{k-1})/(iq_k + q_{k-1})$, and since intermediate fractions are ever-increasing or ever-decreasing with respect to the parameter i, the smallest value of i which will create an intermediate fraction potentially within $[l, r]$ can be directly calculated. Only the intermediate fraction formed with this calculated value of i needs to be tested for membership in $[l, r]$.

Let l_N and l_D be the numerator and denominator of l, and let r_N and r_D be the numerator and denominator of r. In the case of k even; $s_k < l < (l+r)/2$ (otherwise s_k would have been identified as $\in [l, r]$, see Algorithm 4.14); $s_{k+1} \geq (l+r)/2$; with increasing i, $(ip_k + p_{k-1})/(iq_k + q_{k-1})$ forms a decreasing sequence; and the inequality we seek to solve is

$$\frac{ip_k + p_{k-1}}{iq_k + q_{k-1}} \leq \frac{r_N}{r_D}. \qquad (4.62)$$

Solving (4.62), the smallest integral value of i that will suffice is

$$i = \left\lceil \frac{r_N q_{k-1} - r_D p_{k-1}}{r_D p_k - r_N q_k} \right\rceil. \qquad (4.63)$$

Similarly, for k odd, the sequence is increasing, and the inequality and solution are

$$\frac{ip_k + p_{k-1}}{iq_k + q_{k-1}} \geq \frac{l_N}{l_D} \rightarrow i = \left\lceil \frac{l_N q_{k-1} - l_D p_{k-1}}{l_D p_k - l_N q_k} \right\rceil. \qquad (4.64)$$

(4.62), (4.63), and (4.64) suggest the following continued fraction algorithm for finding a rational number with the smallest denominator in an interval $[l, r]$.

Algorithm 4.14:

- Calculate all partial quotients a_k and all convergents $s_k = p_k/q_k$ of the midpoint of the interval, $(l+r)/2$.

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[10]Thanks to David M. Einstein [39] and David Eppstein [24] for this observation, contributed via the `sci.math` newsgroup [33], which is the linchpin of Algorithm 4.14.

[11]Regrettably, at this point the cart comes before the horse—the insight and algorithms which follow are based on continued fractions, which are not covered until Chapter 5, *Continued Fractions And Related Topics*. We apologize for the potential necessity of reading this work out of order.
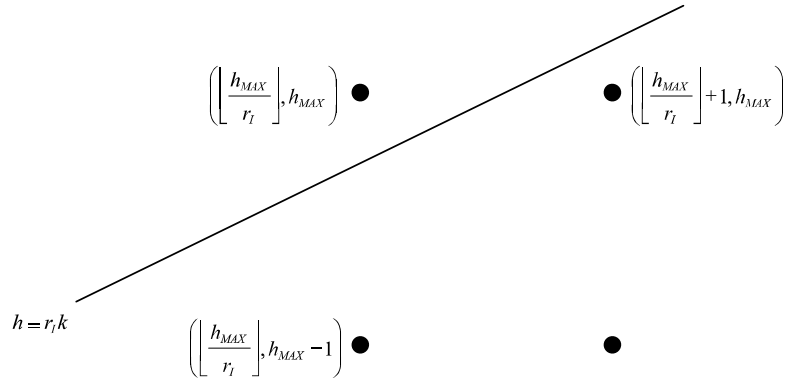
Figure 4.4: Graphical Interpretation Of Case II: $h_{MAX}/k_{MAX} < r_I < 1$

- For each convergent $s_k = p_k/q_k$, in order of increasing k:

  - If $s_k = p_k/q_k \in [l, r]$, s_k is a rational number with the lowest denominator, STOP.
  - If k is even,
    - Calculate i according to (4.63). If $i < a_{k+1}$ and the intermediate fraction $(ip_k + p_{k-1}) / (iq_k + q_{k-1}) \geq l$, this intermediate fraction is a rational number with the lowest denominator, STOP.
  - Else if k is odd,
    - Calculate i according to (4.64). If $i < a_{k+1}$ and the intermediate fraction $(ip_k + p_{k-1}) / (iq_k + q_{k-1}) \leq r$, this intermediate fraction is a rational number with the lowest denominator, STOP.

Algorithm 4.14 is approximately $O(log\ k_{MAX})$, since there are a fixed number of steps per convergent, and the maximum number of convergents is $O(log\ k_{MAX})$. Once a rational number with the smallest denominator q_{MIN} is located, (4.60) can be applied to bound $|r_A - r_I|$; namely,

$$\left| \frac{h}{k} - r_I \right| < \frac{1}{2q_{MIN}\ max(q_{MIN}, k_{MAX} - q_{MIN})}. \tag{4.65}$$

Case II: $h_{MAX}/k_{MAX} < r_I < 1$

If $h_{MAX}/k_{MAX} < r_I < 1$, a graphical argument (Figure 4.4) can be used to more tightly bound the maximum distance between terms of $F_{k_{MAX}, \overline{h_{MAX}}}$.

In this case, a formable term at or to the left[12] of r_I is represented by the point $(\lfloor h_{MAX}/r_I \rfloor + 1, h_{MAX})$ in the integer lattice, and a formable term at or

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[12]To the left on the number line, but to the right in Figure 4.4.

to the right of r_I is represented by the point $(\lfloor h_{MAX}/r_I \rfloor, h_{MAX})$ in the integer lattice. Thus, the maximum distance between neighboring terms in $F_{k_{MAX}, \overline{h_{MAX}}}$ is given by the difference of these two terms,

$$\frac{h_{MAX}}{\left\lfloor \frac{h_{MAX}}{r_I} \right\rfloor} - \frac{h_{MAX}}{\left\lfloor \frac{h_{MAX}}{r_I} \right\rfloor + 1} = \frac{h_{MAX}}{\left\lfloor \frac{h_{MAX}}{r_I} \right\rfloor^2 + \left\lfloor \frac{h_{MAX}}{r_I} \right\rfloor}, \qquad (4.66)$$

and the maximum distance from r_I to a neighboring term is given by

$$\left| \frac{h}{k} - r_I \right| \leq \frac{h_{MAX}}{2 \left(\left\lfloor \frac{h_{MAX}}{r_I} \right\rfloor^2 + \left\lfloor \frac{h_{MAX}}{r_I} \right\rfloor \right)}. \qquad (4.67)$$

Note that Case II will exist only if $h_{MAX}/k_{MAX} < 1$.

Case III: $1 < h_{MAX}/k_{MAX} < r_I$

It can be established graphically, using the coordinate system of Figure 4.1, Figure 4.2, or Figure 4.3, that the line $h = r_I k$ intercepts the line $h = h_{MAX}$ at the point $(h_{MAX}/r_I, h_{MAX})$. It is clear from a graphical argument that all of the terms of the Farey series of order $\lfloor h_{MAX}/r_I \rfloor$ are available as neighbors of r_I. Therefore,

$$\left| \frac{h}{k} - r_I \right| \leq \frac{1}{2 \left\lfloor \frac{h_{MAX}}{r_I} \right\rfloor}. \qquad (4.68)$$

4.9 Acknowledgements

4.10   Exercises

[**4.1**] Prove that Theorem 4.2 holds in the degenerate cases where $h = 1$ and where $k = 1$.

[**4.2**] Prove that Theorem 4.2 holds $\forall i \in \mathbb{Z}$ (rather than $\forall i \in \mathbb{Z}^+$) using the slightly amended notion of reducibility that h/k is irreducible iff $\lfloor h \rfloor / k$ is irreducible.

[**4.3**] In Section 4.4 and Algorithm 4.12 it is stated that for $i \in \mathbb{Z}^+$, $(iN-1)/N$, $i/1$, and $(iN+1)/N$ are consecutive terms in the Farey series or order N, F_N. Prove that $(iN - 1)/N$ and $(iN + 1)/N$ are irreducible, and the left and right Farey neighbors to $i/1$.

[**4.4**] Prove that in F_N the maximum distance between terms $1/N$ can occur only adjacent to an integer.

Chapter 5

Continued Fractions And Related Topics

"I began by saying that there is probably less difference between the positions of a mathematician and a physicist than is generally supposed, and that the most important seems to me to be this, that the mathematician is in much more direct contact with reality ... mathematical objects are so much more what they seem. A chair or a star is not in the least what it seems to be; the more we think of it, the fuzzier its outlines become in the haze of sensation which surround it; but '2' or '317' has nothing to do with sensation, and its properties stand out the more clearly the more closely we scrutinize it."

—G. H. Hardy [29], pp. 128-130

5.1   Introduction

A *finite simple continued fraction* is a fraction of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ldots + \cfrac{1}{a_n}}}} = [a_0; a_1, a_2, \ldots, a_n], \qquad (5.1)$$

where $a_0 \in \mathbb{Z}^+$ and $a_i \in \mathbb{N}$, $i > 0$. Each integer a_i is called an *element* or *partial quotient* of the continued fraction. We require, except in the case of the

continued fraction representation of an integer, that the final element a_n not be equal to 1.[1]

Continued fractions are quite unwieldly to write and typeset, and so a continued fraction in the form of (5.1) is written as $[a_0; a_1, a_2, \ldots, a_n]$. Note that the separator between a_0 and a_1 is a semicolon ('; '), and that all other separators are commas (', '). In some works, commas are used exclusively; and in other works, the first element is a_1 rather than a_0. Throughout this work, the notational conventions illustrated in (5.1) are followed.

In this chapter, the framework of continued fractions is presented in the context of finding rational numbers in F_N, the Farey series of order N, enclosing an arbitrary $r_I \in \mathbb{R}^+$. The continued fraction algorithm presented (Algorithm 5.26) is $O(log N)$, and so is suitable for finding the best rational approximations in F_N even when N is very large. Because our emphasis is on practical applications rather than number theory, we don't include more information than is necessary to understand the applications we have in mind.

The study of continued fractions is a topic from number theory (a branch of mathematics). It may be counterintuitive to anyone but a number theorist that continued fractions can be used to economically find best rational approximations, or that continued fractions are anything but a parlor curiosity. C.D. Olds ([61], p. 3) comments:

> At first glance, nothing seems simpler or less significant than writing a number, for example $\frac{9}{7}$, in the form
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> $$\frac{9}{7} = 1 + \frac{2}{7} = 1 + \frac{1}{\frac{7}{2}} = 1 + \frac{1}{3 + \frac{1}{2}} = 1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{1}}}. \qquad (5.2)$$
>
> It turns out, however, that fractions of this form, called "continued fractions", provide much insight into mathematical problems, particularly into the nature of numbers.
>
> Continued fractions were studied by the great mathematicians of the seventeenth and eighteenth centuries and are a subject of active investigation today.

5.2   History Of Continued Fractions

The only work we are aware of that explicitly treats the history of continued fractions is [57]. Although the history of continued fractions is complex, two points are clear. First, it is clear that Euclid's GCD algorithm (Algorithm 5.15), which was known no later than around 300 B.C., represents the historical origin

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[1]The reason for this restriction is discussed later.

of the continued fraction. Second, it is clear that the utility of the apparatus of continued fractions in finding best rational approximations—specifically the properties of convergents—was understood by the 17th century.

In this section, we present some excerpts from [57] which show the very early use of continued fractions to obtain best rational approximations with a numerator and denominator less than certain prescribed limits. We simply demonstrate that the technique we present was known by the 17th century (with the possible exception of the second component of Theorem 5.20), and we don't attempt to describe the other uses of continued fractions or the significance of continued fractions in mathematics or number theory.

Although we present best rational approximations in the context of being able to effectively use processor integer multiplication and division instructions, earlier historical work was aimed at either providing rational approximations to irrational numbers ($\sqrt{2}$ or π, for example), or at determining optimal numbers of gear teeth (in mechanical systems). Naturally, the need for best rational approximations in the context of computer arithmetic is a relatively recent development.

In the introduction of [57], Brezinski hints at the broad application and importance of continued fractions:

> The history of continued fractions is certainly one of the longest among those of mathematical concepts, since it begins with Euclid's algorithm for the greatest common divisor at least three centuries B.C. As it is often the case and like Monsieur Jourdain in Molière's "le bourgeois gentilhomme" (who was speaking in prose though he did not know he was doing so), continued fractions were used for many centuries before their real discovery.

> The history of continued fractions and Padé approximants is also quite important, since they played a leading role in the development of some branches of mathematics. For example, they were the basis for the proof of the transcendence of π in 1882, an open problem for more than two thousand years, and also for our modern spectral theory of operators. Actually they still are of great interest in many fields of pure and applied mathematics and in numerical analysis, where they provide computer approximations to special functions and are connected to some convergence acceleration methods. Continued fractions are also used in number theory, computer science, automata, electronics, etc. . . .

Notice that Theorem 5.20 has two components. First, it is shown that the highest-order convergent with an acceptable denominator is closer to a/b than any Farey neighbor to this convergent (thus, this convergent must be either a left or right Farey neighbor of a/b). Second, it is shown what the other Farey neighbor must be. It is historically clear that the properties of convergents as best rational approximations were understood by the 17th century (this is the

first part of Theorem 5.20). However, it is not historically clear when the *second* part of Theorem 5.20 was discovered.

Even in Khinchin's classic work, [59], Theorem 15, p. 22, Khinchin stops just short of the result presented as the second part of Theorem 5.20. Khinchin writes:

> THEOREM 15. *Every best approximation of a number is a convergent or an intermediate fraction of the continued fraction representing that number.*

Theorem 5.20 goes slightly farther than Khinchin's *THEOREM 15*, above. Khinchin states that a best approximation will be a convergent or an intermediate fraction—but Theorem 5.20 goes slightly farther to indicate *exactly which* intermediate fraction is potentially the best approximation. Khinchin's *THEOREM 15* is correct, but could be strengthened. Khinchin's work was first published in 1935. This raises the [unlikely] possibility that the second part of Theorem 5.20 had not been published even as recently as 1935, although we (the authors) don't have the ability to confirm or refute this.

In [57], p. 70, Brezinski writes:

> In the same period, algorithms equivalent to continued fractions were still used to find approximate values for ratios and to simplify fractions. We have already mentioned Albert Girard.

> Among the other authors who treated the subject, the most prominent is Daniel SCHWENTER (Nürnberg, 31.1.1585 - Altdorf, 19.1.1636), who wrote two books "*Geometriae practicae novae et auctae tractatus*" published in 1627 and "*Delicae Physico-mathematicae*" which appeared in 1636 followed by a second edition in 1651.

> In his first book, Schwenter found approximations of 177/233 by finding their g.c.d. and gave the successive convergents 79/104, 19/25, 3/4, 1/1, and 0/1. His calculations were arranged in a table[2] ... although he gave no explanation of the method.

On p. 84, Brezenski writes:

> Wallis also made use of continued fractions in his book "*A treatise of algebra both historical and practical*" (published in 1685), to approximate ratios with large numerators and denominators:

> "*Before I leave the business of Decimal Parts, and the advantages which in practice may there cause; I have thought fit here to insert a Process Of Reducing Fractions or Proportions to smaller termes, retaining as near as may be, the just value.*

> *It was occasion'd by a Problem sent me (as I remember) about the Year 1663 or 1664, by Dr. Lamplugh the present Bishop of Exeter*

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[2]The table is reproduced in [57], but is omitted here.

from (his Wives Father) Dr. Davenant then one of the Prebends Residentaries of the Church of Salisbury, a very worthy Person, of great Learning and Modesty; as I mire inderstand from persons well acquainted with him, and by divers Writings of his which I have seen, though I never had the opportunity of being personally acquainted with him, otherwise than by Letter. And amongst his other Learning, he was very well skilled the Mathematicks, and a diligent Proficient therein.

He sent me (as it abovesaid) a Fraction (which what it was I do not now particulary remember) who's Numerator and Denominator were, each of them of about six or seven places; and Proposed to find the nearest Fraction in value to it, whose Denominator should not be greater than 999."

The Problem

A Fraction (or Proportion) being assigned, to sind one as near as may be equal to it, in Numbers non exceeding a Number given, and in the smallest Terms.

As (for instance), the Fraction $\frac{2684769}{8376571}$ (or the Proportion of 2684769 to 8376571) being assigned, to sind one equal to it (if it may be) or at least the next Greater, or the next Lesser, which may be expressed in Numbers not greater than 999; that is, in numbers not exceeding three places.

If the Fraction sought (whose terms are not to be greater than a Number given) be the Next Greater than a Fraction Proposed; divide the proposed Fractions Denominator by its Numerator: If the Next-Lesser, then the Numerator by the Denominator, continuing the Quotient in Decimal Parts, to such an Accuracy as shall be sufficient; which Quotient for the Next-Greater, is to be the Denominator answering to the Numerator 1: But for the next Lesser, it is to be the Numerator answering to the Denominator 1: Completing a Fraction as near as shall be necessary to that Proposed, which Fraction I call to First Fraction Compleat: And the same wanting the Appendage of Decimal parts, I call, the First Fraction Cartail'd.

Khen by this Appendage of the First Fraction, divide 1 Integer, and by the Integer Number which is Next-Less then the sull Quotient, (that is, in case such Quotient be just an Interger Number, by the

Integer Next-Less than it; but is it be an Interger with Decimal parts annexed, than by that Integer without those

Decimal parts;) multiply both Terms of the first Fraction Compleat, (the Numerator and the Denominator;) And the Products of such Multiplication, I call the Continual Increments of those Terms respectively. And so much as the Appendage of Decimal parts in such Continual Increment wants of 1 Integer, I call the Complements of the Appendage of the continual Increment.

Then both to the Numerator and the Denominator of the First Fraction, add (respectively) its continual Increment, which make the Terms of the Second Fraction; and these again (respectively) increased by the same Continual Increment, make the Terms of the Third Fraction: And so onward, as long as the Fraction so arising hath an Appendage, which is not less than the Complement of the Appendage of the Continual Increment.

But where such Appendage becomes less than that Complement, that Fraction I call the Last of the First Order; which also is to be the First of the Second Order."

Although Wallis' archaic English above is difficult to decipher, it appears that Wallis is describing the process of obtaining the convergents and intermediate fractions of the continued fraction representation of a rational number.

On p. 86, Brezenski writes:

We have already mentioned the Dutch mathematician and astronomer Christiaan HUYGENS (The Hague, 14.4.1629 - The Hague, 8.6.1695). He made several contributions to continued fractions and related matters.

In 1682, Huygens built an automatic planetarium. To this end, he used continued fractions, as described in his book "*Descriptio automati planetarii*", which was published after his death (The Hague, 1698). In one year the earth covers $359° \ 45' \ 40'' \ 30'''$ and Saturn $12° \ 13' \ 34'' \ 18'''$, which gives the ratio 77708431/2640858.

For finding the smallest integers whose ratio is close to the preceding one, he divided the greatest number by the smallest, then the smallest by the first remainder, and so on, which is Euclid's algorithm. He thus got

$$29 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{4 + \ldots}}}}}}$$

for the ratio.

The fourth convergent of this continued fraction is 206/7, which gave him the number of teeth for the gears of his planetarium, only producing an error of $40'$ in a century! [H. 177], [H. 272].

In a work, undated but not after 1687, he treats the general problem:

"Etant donnés deux grands nombres ayant entr'eux un certain rapport, en trouver d'autres plus petits pour les dents des roues qui ne soient pas incommodes par leurs grandeurs et qui aient entr'eux à peu près le même rapport, de telle facon qu'aucun couple de nombres plus petits ne fournisse un rapport plus approchant de la vraie valeur."[3]

Thus Huygens was conscious of the property of best approximation exhibited by continued fractions. He explained his method as follows:

"Pour trouver donc des nombres plus petits qui expriment approximativement ce rapport, je divise le plus grand des nombres par le plus petit, puis le plus petit par le reste de la première division et ensuite ce reste par le noveau reste ... Poursuivant ce calcul aussi longtemps que possible, on parvient enfin par la division à un reste 1."[4]

Then he makes the following comments:

"Or, lorsqu'on néglige à partir d'une fraction quelconque les derniers termes de la série et celles qui la suivent, et qu'on réduit les autres plus le nombre entier à un commun dénominateur, le rapport de ce dernier au numérateur, sera voisin de celui du plus petit nombre

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[3]English translation [103]: *If we consider two large numbers forming a given ratio, we need to find another set of smaller numbers for the teeth of the gearwheels, which are not inconvenient in their size and which bear the very same ratio between them, in such a way that no other pair of smaller numbers brings a ratio closer to the actual value.*

[4]English translation [103]: *Thus to find some smaller numbers that approximately express this ratio, I divide the largest of the numbers by the smallest, then the smallest by the remainder of the first division and then this remainder by the new remainder, continuing this calculation as long as possible, we finally end up with a division into a remainder of 1.*

donné au plus grand; et la différence sera si faible qu'il serait impossible d'obtenir un meilleur accord avec des nombres plus petits."[5]

He proves this result and applies it to the continued fraction for π.

Let us give the opinion of the French astronomer Jean Baptiste Joseph DELAMBRE (Amiens, 19.9.1749 - Paris, 19.8.1822), about this part of Huygens' work. It is quite interesting [H. 108]:

" ... ; enfin il décrit son planétaire.[6]

Ces sortes de machines ne sont que des objets de curiosité pour les amateurs, ils sont absolument inutiles à l'Astronomie; celle d'Huygens était destinée à montrer les mouvements elliptiques des planètes, suivant les idées de Képler. Le problème à résoudre était celui-ci: Etant donné deux grands nombres, trouver deux autres nombres plus pitits et plus commodes, qui soient à peu près dans la même raison. Il y emploie les fractions continues, et sans donner la théorie analytique de ces fractions, il les applique à des exemples. Il trouve ainsi le nombre des dents qui'il convient de donner aux roues.[7]

Cette propriété des fractions continues, paraît à Lagrange, une des principales découvertes d'Huygens. Cet éloge un peu exagéré fut sans doute dicté à Lagrange par l'usage qu'il a su faire de ces fractions dans l'Analyse. Quelques géomètres ont paru douter des avantages de ces fractions et de l'utilité qu'elles peuvent avoir dans les recherches analytiques. Quant au problème des rouages, il nous semble qu'on peut le résoudre d'une manière plus simple et plus commode par l'Arithmétique ordinaire. Nous avons déjà appliqué notre méthode aux intercalations du calendrier. Nous allons l'appliquer aux deux exemples choisis par Huyhens."[8]

Delambre concludes:

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[5]English translation [103]: *However, when, from an ordinary fraction, we neglect the last terms of the run and the ones that follow, and when we reduce the others plus the integer to a common denominator, the ratio of the latter to the numerator will be in the neighborhood of the smallest given number to the largest; and the difference will be so small that it would be impossible to obtain a better tune with some smaller numbers.*

[6]English translation [103]: *... ; finally, he describes his planetarium.*

[7]English translation [103]: *These kinds of machines are mere objects of curiosity for the amateurs, completely useless to astronomy; Huygens' machine was meant to demonstrate the elliptic movements of the planets, following Kepler's ideas. The problem to solve was the following: given two large numbers, we need to find two other numbers, smaller and more convenient, which are more or less in the same ratio. To achieve this, Huygens uses continued ratios, and, without giving the analytic theory of these ratios, he applies it to some examples. Thus, he is able to determine the number of teeth needed for the gearwheels.*

[8]English translation [103]: *The property of continued fractions seems, to Lagrange, one of the main discoveries of Huygens. This slightly overdone praise was probably induced in Lagrange for the use that he made of the fractions in his Analysis. Some surveyors seemed to have questioned the advantages of these fractions and their use in analytical research. As far as the gearing problem is concerned, it seems to us that we can solve it in a simpler and easier way with ordinary arithmetic. We have already applied our methodology to the intercalation of the calendar. We are going to apply it to the two examples chosen by Huygens.*

"Les fractions continues ne m'ont jamais paru qu'une chose curieuse qui, au reste, ne servait qu'à obscurcir et compliquer et je n'en ai jamais fait d'usage que pour m'en démontrer l'inutilité."[9]

This was not a prophetic view!

Thus, it is clear that the use of continued fraction convergents as best rational approximations dates back to at least the 17th century (this is the first part of Theorem 5.20). However, the details of the historical appearance of the second part of Theorem 5.20 (the formula for the other Farey neighbor, Eq. 5.71) are not known to the authors.

5.3 Overview Of The Apparatus

The apparatus of continued fractions is best viewed as an alternate apparatus for representing real numbers. Knowledge of the first n partial quotients of the continued fraction representation of a real number x is equivalent to the knowledge that the number lies in a certain partition (Eqns. 5.3, 5.4, and 5.5). With additional partial quotients, the partitions become more restrictive.

$$(x = [a_0] \lor x = [a_0; \dots]) \leftrightarrow (a_0 \leq x < a_0 + 1) \tag{5.3}$$

$$(x = [a_0; a_1] \lor x = [a_0; a_1, \dots]) \leftrightarrow \left(a_0 + \frac{1}{a_1 + 1} < x \leq a_0 + \frac{1}{a_1} \right) \tag{5.4}$$

$$(x = [a_0; a_1, a_2] \lor x = [a_0; a_1, a_2, \dots])$$
$$\updownarrow$$
$$\left(a_0 + \frac{1}{a_1 + \dfrac{1}{a_2}} \leq x < a_0 + \frac{1}{a_1 + \dfrac{1}{a_2 + 1}} \right) \tag{5.5}$$

Algorithms for finding the continued fraction representation of a real number are best viewed as algorithms for determining in which partition a real number lies. However, what is special (for our purposes) is that the partitions imposed by the apparatus of continued fractions have a special relationship with best rational approximations—namely, that all numbers (both rational and irrational) with the same partial quotients up to a point also have the same Farey neighbors up to a certain order. Stated more colloquially, the apparatus of continued fractions hacks up the real number line in a way that is especially meaningful for finding best rational approximations.

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[9]English translation [103]: *Continued fractions never appeared to me as something more than a mere curiosity that, at the end of the day, only serves to darken and complicate matters, and I only used them to demonstrate their uselessness.*

Table 5.1: Continued Fraction Partial Quotients Of 67/29 (Example 5.2)

Index (k)	$dividend_k$	$divisor_k$	a_k	$remainder_k$
-1	N/A	67	N/A	29
0	67	29	2	9
1	29	9	3	2
2	9	2	4	1
3	2	1	2	0

5.4 Continued Fraction Representation Of Rational Numbers

Without proof, we present the following algorithm, Algorithm 5.1, for determining the continued fraction representation (i.e. the partial quotients) of a non-negative rational number a/b.

Algorithm 5.1 (Continued Fraction Representation Of A Rational Number a/b):

- $k := -1$.

- $divisor_{-1} := a$.

- $remainder_{-1} := b$.

- Repeat

    - $k := k + 1$.
    - $dividend_k := divisor_{k-1}$.
    - $divisor_k := remainder_{k-1}$.
    - $a_k := dividend_k \ div \ divisor_k$.
    - $remainder_k := dividend_k \ mod \ divisor_k$.

- Until ($remainder_k = 0$).

Example 5.2: Find the continued fraction partial quotients of 67/29.[10]
Solution: Table 5.1 shows the application of Algorithm 5.1 to find the contin-

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[10]This example is reproduced from Olds [61], p. 8.

ued fraction partial quotients of 67/29. From Table 5.1, the continued fraction representation of 67/29 is $[2; 3, 4, 2]$.

The process of obtaining the continued fraction representation of a rational number is a process of obtaining each partial quotient a_i, and then processing the remainder at each step to obtain further partial quotients. Noting that the dividend and divisor at each step come from previous remainders (except for $k = 0$ and $k = 1$) allows us to simplify notation from Algorithm 5.1. If r_i is used to denote the remainder from the division that produced a_i, the following recursive equations come immediately.

$$\frac{a}{b} = a_0 + \frac{r_0}{b} = a_0 + \frac{1}{\frac{b}{r_0}}, \ 0 < r_0 < b \tag{5.6}$$

$$\frac{b}{r_0} = a_1 + \frac{r_1}{r_0}, \ 0 < r_1 < r_0 \tag{5.7}$$

$$\frac{r_0}{r_1} = a_2 + \frac{r_2}{r_1}, \ 0 < r_2 < r_1 \tag{5.8}$$

Finally, nearing the termination of Algorithm 5.1:

$$\frac{r_{n-3}}{r_{n-2}} = a_{n-1} + \frac{r_{n-1}}{r_{n-2}}, \ 0 < r_{n-1} < r_{n-2} \tag{5.9}$$

$$\frac{r_{n-2}}{r_{n-1}} = a_n \tag{5.10}$$

A natural question to ask is whether Algorithm 5.1 will always terminate—that is, whether we can always find a continued fraction representation of a rational number. We present this result as Lemma 5.3.

Lemma 5.3: Algorithm 5.1 will always terminate: that is, every rational number has a finite continued fraction representation $[a_0; a_1, \ldots, a_n]$.
Proof: Note in Algorithm 5.1 and in (5.6) through (5.10) that the remainder of one round becomes the divisor of the next round, hence the remainders must form a decreasing sequence

$$r_0 > r_1 > r_2 > \ldots > r_{n-2} > r_{n-1}, \tag{5.11}$$

because in general a remainder must be less than the divisor in the division that created it. ■

5.5   Convergents And Intermediate Fractions

Lemma 5.3 shows that every rational number has a finite continued fraction representation. A second reasonable question to ask is whether every finite simple continued fraction corresponds to a rational number. The most convincing way to answer that question would be to devise a concrete procedure for [re-]constructing a rational number from its continued fraction representation.

Given a finite continued fraction $[a_0; a_1, \ldots, a_n]$, it is obvious that a rational number can be constructed using the same algebraic technique that would be applied by hand. Such a technique involves "reconstruction from the right" because we would begin by using a_n and then work backwards to a_0. We illustrate the most obvious technique with an example.

Example 5.4:   Find a rational number a/b corresponding to the continued fraction $[2; 3, 4, 2]$.

Solution: The most obvious technique is to write out the continued fraction and then to algebraically simplify the continued fraction from the bottom up (this is what we call "working from the right", as we begin with a_n). (5.12) through (5.14) illustrate this technique.

$$[2; 3, 4, 2] = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}} \tag{5.12}$$

$$[2; 3, 4, 2] = 2 + \cfrac{1}{3 + \cfrac{2}{9}} \tag{5.13}$$

$$[2; 3, 4, 2] = 2 + \frac{9}{29} = \frac{67}{29} \tag{5.14}$$

Although converting a continued fraction $[a_0; a_1, \ldots, a_n]$ to a rational number working "from the right" is the most intuitively obvious technique because it mirrors how a continued fraction would most naturally be simplified by hand, it is also possible to convert a continued fraction to a rational number "from the left". In all subsequent discussions we embrace the "from the left" technique because it allows us to more economically calculate *convergents*, which have special properties, and which we now describe.

The *kth order convergent* of a continued fraction $[a_0; a_1, \ldots, a_n]$ is the irreducible rational number corresponding to $[a_0; a_1, \ldots, a_k]$, $k \leq n$. In other words, the kth order convergent is the irreducible rational number corresponding to the first $k + 1$ partial quotients of a continued fraction.[11]

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[11] "$k + 1$" because the notational numbering for partial quotients starts at 0 rather than 1.

An nth order continued fraction $[a_0; a_1, \ldots, a_n]$ has $n+1$ convergents, $[a_0]$, $[a_0; a_1]$, $\ldots$, and $[a_0; a_1, \ldots, a_n]$. We denote the kth order convergent as s_k, with numerator p_k and denominator q_k.

Example 5.5: Find all convergents of $[2; 3, 4, 2]$.
Solution: [12]

$$s_0 = [a_0] = [2] = 2 = \frac{2}{1} = \frac{p_0}{q_0} \tag{5.15}$$

$$s_1 = [a_0; a_1] = [2; 3] = 2 + \frac{1}{3} = \frac{7}{3} = \frac{p_1}{q_1} \tag{5.16}$$

$$s_2 = [a_0; a_1, a_2] = [2; 3, 4] = 2 + \cfrac{1}{3 + \cfrac{1}{4}} = \frac{30}{13} = \frac{p_2}{q_2} \tag{5.17}$$

$$s_3 = [a_0; a_1, a_2, a_3] = [2; 3, 4, 2] = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}} = \frac{67}{29} = \frac{p_3}{q_3} \tag{5.18}$$

We now move on to the question of how to convert a continued fraction to a rational number "from the left". We present the canonical algorithm for construction of convergents "from the left". In addition to producing irreducible rational numbers (we prove this property later), the algorithm is convenient because it is economical—lower-order convergents are used in the calculation of higher-order convergents and there are no wasted calculations.

Theorem 5.6 (Rule For Canonical Construction Of Continued Fraction Convergents): The numerators p_i and the denominators q_i of the ith convergent s_i of the continued fraction $[a_0; a_1, \ldots, a_n]$ satisfy the equations

$$p_i = a_i p_{i-1} + p_{i-2} \tag{5.19}$$
$$q_i = a_i q_{i-1} + q_{i-2} \tag{5.20}$$

with the initial values

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[12]Canonically, it is required that all convergents be irreducible. Any valid method can be used to calculate convergents—including algebraic simplification—so long as the rational numbers obtained are irreducible.

$$p_0 = a_0, \qquad p_1 = a_0 a_1 + 1, \tag{5.21}$$

$$q_0 = 1, \qquad q_1 = a_1. \tag{5.22}$$

Proof: [13] The proof is inductive. First, the case of $i = 2$ is verified, then an inductive step is used to show that the theorem applies for $i \geq 3$.

To create a canonical form, we assign $s_0 = [a_0] = p_0/q_0 = a_0/1$. Thus, in all cases, $p_0 = a_0$ and $q_0 = 1$. Similarly, to create a unique canonical form,

$$s_1 = [a_0; a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{p_1}{q_1}, \tag{5.23}$$

and canonically, $p_1 = a_0 a_1 + 1$ and $q_1 = a_1$.

For $i = 2$, we need to verify that the algebraic results coincide with the claims of the theorem. Simplifying s_2 algebraically leads to

$$s_2 = [a_0; a_1, a_2] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = a_0 + \cfrac{1}{\cfrac{a_1 a_2 + 1}{a_2}} = a_0 + \cfrac{a_2}{a_1 a_2 + 1}$$

$$= \frac{a_0(a_1 a_2 + 1) + a_2}{a_1 a_2 + 1} = \frac{a_2(a_0 a_1 + 1) + a_0}{a_2 a_1 + 1}. \tag{5.24}$$

On the other hand, applying the recursive formula claimed by the theorem (Eqns. 5.19, 5.20) yields

$$s_2 = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} = \frac{a_2(a_0 a_1 + 1) + a_0}{a_2(a_1) + 1}, \tag{5.25}$$

which, on inspection, is consistent with the results of (5.24).

We now prove the inductive step. Assume that the recursive relationships supplied as (5.19) and (5.20) hold up through some $s_k = p_k/q_k$. We would like to show that (5.19) and (5.20) then hold for s_{k+1}.

s_k is a fraction of the form

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[13]Reproduced nearly verbatim from [61], Theorem 1.3, pp. 21-23.

$$s_k = [a_0; a_1, a_2, \ldots, a_k] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ldots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}}}. \quad (5.26)$$

In order to form s_{k+1}, note that we can replace a_k by $a_k + 1/a_{k+1}$. (Note that there is no requirement in Eqns. 5.19, 5.20, or 5.24 or 5.26 that the partial quotients a_i be integers.) In other words, we can form a kth order continued fraction having the same value as a $k+1$th order continued fraction by substituting $a_k := a_k + \frac{1}{a_{k+1}}$. Using this substitution we can calculate s_{k+1} using the same recursive relationship shown to be valid in calculating s_k:

$$s_{k+1} = \cfrac{\left(a_k + \cfrac{1}{a_{k+1}}\right)p_{k-1} + p_{k-2}}{\left(a_k + \cfrac{1}{a_{k+1}}\right)q_{k-1} + q_{k-2}} = \frac{(a_k a_{k+1} + 1)p_{k-1} + a_{k+1}p_{k-2}}{(a_k a_{k+1} + 1)q_{k-1} + a_{k+1}q_{k-2}}$$

$$= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} \quad (5.27)$$

Now, we can use the assumption that the recursive relationships hold for s_k, i.e.

$$p_k = a_k p_{k-1} + p_{k-2} \quad (5.28)$$
$$q_k = a_k q_{k-1} + q_{k-2} \quad (5.29)$$

Substituting (5.28) and (5.29) into (5.27) yields

$$s_{k+1} = \frac{p_{k+1}}{q_{k+1}} = \frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}}. \quad (5.30)$$

This completes the inductive step and the proof. ∎

Remarks: Note that this algorithm gives a way to convert a continued fraction $[a_0; a_1, \ldots, a_n]$ to a rational number a/b, as the value of a continued fraction is the value of the final convergent s_n. Note also that it is possible to convert a continued fraction to a rational number starting from a_n (i.e. working "from the right"), and that starting with a_n is probably the more intuitive approach.

It is sometimes convenient to consider a convergent of order -1 ([59], p. 5), and for algebraic convenience to adopt the convention that $p_{-1} = 1$ and

$q_{-1} = 0$. If this is done, the recursive relationships of Theorem 5.6 apply for $k \geq 1$ rather than for $k \geq 2$. All of the subsequent theorems and proofs assume this convention.

   We now prove several properties of convergents.

Theorem 5.7:   For all $k \geq 0$,

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k \tag{5.31}$$

Proof: [14] Multiplying (5.19) by p_{k-1}, multiplying (5.20) by q_{k-1}, then subtracting the equations yields

$$q_k p_{k-1} - p_k q_{k-1} = -(q_{k-1} p_{k-2} - p_{k-1} q_{k-2}), \tag{5.32}$$

and since $q_0 p_{-1} - p_0 q_{-1} = 1$, the theorem is proved.   ∎

Corollary I: For all $k \geq 1$,

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}. \tag{5.33}$$

Proof: (5.33) can be obtained in a straightforward way by algebraic operations on (5.31).   ∎

Theorem 5.8:   For all $k \geq 1$,

$$q_k p_{k-2} - p_k q_{k-2} = (-1)^{k-1} a_k. \tag{5.34}$$

Proof: [15] By multiplying (5.19) by q_{k-2} and (5.20) by p_{k-2} and then subtracting the first from the second, we obtain, on the basis of Theorem 5.7,

$$q_k p_{k-2} - p_k q_{k-2} = a_k(q_{k-1} p_{k-2} - p_{k-1} q_{k-2}) = (-1)^{k-1} a_k, \tag{5.35}$$

which completes the proof.   ∎

   The results in Theorems 5.7 and 5.8 allow us to establish the relative ordering of convergents. Theorems 5.7 and 5.8 demonstrate that even-ordered convergents form an increasing sequence and that odd-ordered convergents form

[14]From [59], Theorem 2, p. 5.
[15]From [59], Theorem 3, p. 6.

a decreasing sequence, and that every odd-ordered convergent is greater than every even-ordered convergent.

Theorem 5.9: For all $k \geq 0$, $s_k = p_k/q_k$ is irreducible.
Proof: This proof comes immediately from the form of (5.31). Without coprimality of p_k and q_k, the difference of ± 1 is impossible (see Lemma 3.5). ∎

Example 5.10: Find an irreducible rational number a/b corresponding to the continued fraction $[2; 3, 4, 2]$.
Solution: Application of Theorem 5.6 yields the following convergents. The final convergent s_3 is the value of the continued fraction $[2; 3, 4, 2]$ and Theorem 5.9 assures us that each convergent is irreducible.

$$p_{-1} = 1, \; q_{-1} = 0 \tag{5.36}$$

$$s_0 = \frac{p_0}{q_0} = \frac{a_0}{1} = \frac{2}{1} \tag{5.37}$$

$$s_1 = \frac{p_1}{q_1} = \frac{a_1 p_0 + p_{-1}}{a_1 q_0 + q_{-1}} = \frac{(3)(2) + (1)}{(3)(1) + (0)} = \frac{7}{3} \tag{5.38}$$

$$s_2 = \frac{p_2}{q_2} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} = \frac{(4)(7) + (2)}{(4)(3) + (1)} = \frac{30}{13} \tag{5.39}$$

$$s_3 = \frac{p_3}{q_3} = \frac{a_3 p_2 + p_1}{a_3 q_2 + q_1} = \frac{(2)(30) + (7)}{(2)(13) + (3)} = \frac{67}{29} \tag{5.40}$$

Note that this result coincides with Example 5.2.

We've shown in Algorithm 5.1 that any rational number can be expressed as a continued fraction, and with Theorem 5.6 that any finite continued fraction can be converted to a rational number. Although we don't say more until Section 5.7, it follows directly that any irrational number results in an *infinite* (or non-terminating) continued fraction, and that any infinite continued fraction represents an irrational number. In the theorems that follow, we don't treat infinite continued fractions with mathematical rigor, because our emphasis is on specific applications of continued fractions.

Theorem 5.11: For a finite continued fraction representation of the [rational] number α, every even-ordered convergent is less than α and every odd-ordered

convergent is greater than α, with the exception of the final convergent s_n, which is equal to α. For an infinite continued fraction corresponding to the [irrational] real number α, every even-ordered convergent is less than α, and every odd-ordered convergent is greater than α.

Proof (Informal):   In the case of a finite continued fraction, the proof is obvious and immediate. Since s_n, the final convergent, is equal to the rational number α, Theorems 5.7 and 5.8 demonstrate this unequivocally.

In the case of an infinite continued fraction, note the form of the proof of Theorem 5.6, where the substitution of $a_k := a_k + 1/a_{k+1}$ is made. It can be demonstrated that for any even-ordered convergent s_k, additional partial quotients (except $a_{k+1} = a_n = 1$, which isn't allowed in general or even possible with an infinite continued fraction) can only increase the value. It can similarly be demonstrated that additional partial quotients can only decrease the value of an odd-ordered convergent. Because the continued fraction is infinite, any particular even-ordered convergent will be increased if more partial quotients are allowed, and any particular odd-ordered convergent will be decreased in value if more partial quotients are allowed. Thus, we can conclude that all even-ordered convergents are less than the value of α, and all odd-ordered convergents are greater than the value of α.[16]

Theorem 5.12:   For $k \geq 2$,

$$q_k \geq 2^{\frac{k-1}{2}}. \tag{5.41}$$

Proof:[17] For $k \geq 2$,

$$q_k = a_k q_{k-1} + q_{k-2} \geq q_{k-1} + q_{k-2} \geq 2q_{k-2}. \tag{5.42}$$

Successive application of this inequality yields

$$q_{2k} \geq 2^k q_0 = 2^k, \ q_{2k+1} \geq 2^k q_1 \geq 2^k, \tag{5.43}$$

which proves the theorem. Thus, the denominators of convergents increase at least as rapidly as the terms of a geometric progression.   ■

Remarks: (1) This minimum geometric rate of increase of denominators of convergents is how we make the claim that Algorithms 5.26 and 5.31 are $O(log\ N)$ and that Algorithms 5.29 and 5.33 are $O(log\ max(h_{MAX}, k_{MAX}))$. (2) This theorem supplies the *minimum* rate of increase, but the demonominators of convergents can increase much faster. To achieve the minimum rate of increase,

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[16]To make this proof more formal would require the discussion of *remainders*, which wouldn't contribute to the applications discussed in this work.
[17]From [59], Theorem 12, p. 13.

every a_k must be 1, which occurs only with the continued fraction representation of $\sqrt{5}/2 + 1/2$ (the famous golden ratio). (See also Exercise 5.4.)

Since Theorem 5.6 provides a concrete procedure for going from a continued fraction $[a_0; a_1, \ldots, a_n]$ to a rational number a/b that, when Algorithm 5.1 is applied, will again result in $[a_0; a_1, \ldots, a_n]$, we have successfully demonstrated that every continued fraction $[a_0; a_1, \ldots, a_n]$ corresponds to [at least one] rational number a/b.

The next natural questions to ask are questions of representation uniqueness and the nature of the mapping between the set of rational numbers and the set of continued fractions. For example, will 32/100 and 64/200 have the same continued fraction representation $[a_0; a_1, \ldots, a_n]$? Do two different continued fractions ever correspond to the same rational number? We answer these questions now.

Algorithm 5.1 will produce the same $[a_0; a_1, \ldots, a_n]$ for any ia/ib, i.e. all rational numbers which are equivalent in value will generate the same continued fraction representation (see Lemma 5.13).

It was hinted in the introduction (Section 5.1, Footnote 1) that, except in the case of representing an integer, it is not allowed for the final partial quotient a_n to be 1. We now explain the reasons why this must be disallowed. First, if $a_n = 1$, then a_{n-1} can be increased by 1 and the continued fraction can be reduced in order by 1 and while still preserving its value. For example, it can easily be verified that $[1; 2, 3, 3, 1]$ and $[1; 2, 3, 4]$ represent the same number. However, this observation alone is not enough to recommend a canonical form—this observation does not suggest that $[1; 2, 3, 4]$ should be preferred over $[1; 2, 3, 3, 1]$. However, what *can* be noted is that that a continued fraction representation with $a_n = 1$, $n > 0$ cannot be attained using Algorithm 5.1 or (5.6) through (5.10), because a form with $a_n = 1$, $n > 0$ violates the assumption that successive remainders are ever-decreasing (see Eq. 5.11). The property that remainders are ever-decreasing is a necessary condition in proofs of some important properties, and so requiring that $a_n \neq 1$, $n > 0$ is the most natural convention for a canonical form.

Lemma 5.13: Algorithm 5.1 will produce the same result $[a_0; a_1, \ldots, a_n]$ for any ia/ib, i.e. a/b need not be reduced before the algorithm is applied.

Proof: Assume that a/b is irreducible, and that ia/ib, $i \in \{2, 3, \ldots\}$ is used as input to the algorithm. By definition, any rational number with the same value as a/b is of the form ia/ib, $i \in \mathbb{N}$. Note that (5.6) through (5.10) "scale up", while still producing the same partial quotients $[a_0; a_1, \ldots, a_n]$. Specifically:

$$\frac{ia}{ib} = a_0 + \frac{ir_0}{ib} = a_0 + \frac{1}{\frac{ib}{ir_0}} \tag{5.44}$$

$$\frac{ib}{ir_0} = a_1 + \frac{ir_1}{ir_0} \tag{5.45}$$

$$\frac{ir_0}{ir_1} = a_2 + \frac{ir_2}{ir_1} \tag{5.46}$$

Finally, nearing the termination of Algorithm 5.1:

$$\frac{ir_{n-3}}{ir_{n-2}} = a_{n-1} + \frac{ir_{n-1}}{ir_{n-2}} \tag{5.47}$$

$$\frac{ir_{n-2}}{ir_{n-1}} = a_n \tag{5.48}$$

Thus, it is easy to show that Algorithm 5.1 will produce the same continued fraction representation regardless of whether the input to the algorithm is reduced. It is also easy to show that the last non-zero remainder as the algorithm is applied (r_{n-1}, in Eqns. 5.47 and 5.48) is the greatest common divisor of ia and ib (this is done in the proof of Algorithm 5.15). ∎

Lemma 5.14: So long as $a_n \neq 1$, $n > 0$, a rational number a/b has only one [unique] continued fraction representation $[a_0; a_1, \ldots, a_n]$.
Proof: Assume that two different continued fractions, $[a_0; a_1, \ldots, a_m]$ and $[\overline{a_0}; \overline{a_1}, \ldots, \overline{a_n}]$, correspond to the same rational number a/b. By *different*, we mean either that $m = n$ but $\exists i, a_i \neq \overline{a_i}$, or that $m \neq n$.

Note that Theorem 5.6 will map from any continued fraction to an irreducible rational number a/b. Assume we apply Theorem 5.6 to $[a_0; a_1, \ldots, a_m]$ to produce a/b, and to $[\overline{a_0}; \overline{a_1}, \ldots, \overline{a_n}]$ to produce $\overline{a}/\overline{b}$. Because two irreducible rational numbers are equal iff their components are equal, $[(a/b) = (\overline{a}/\overline{b})] \rightarrow [(a = \overline{a}) \wedge (b = \overline{b})]$. Because $a/b = \overline{a}/\overline{b}$, we denote both of these numbers simply as a/b.

By (5.50), $a = a_0 b + r_0 = \overline{a_0} b + \overline{r_0}$, $0 < r_0, \overline{r_0} < b$. Because $r_0, \overline{r_0} < b$, there is only one combination of a_0 and r_0 or of $\overline{a_0}$ and $\overline{r_0}$ that can result in a. Thus, we can conclude that $a_0 = \overline{a_0}$ and $r_0 = \overline{r_0}$. This type of reasoning, can be carried "downward" inductively, each time fixing a_i and r_i. Finally, we must conclude that $(a/b = \overline{a}/\overline{b}) \rightarrow [a_0; a_1, \ldots, a_m] = [\overline{a_0}; \overline{a_1}, \ldots, \overline{a_n}]$ and that $m = n$. ∎
Remarks: The case of $a_n = 1$, $n > 0$ deserves further discussion. Theorem 5.6 will produce an irreducible rational number even if $a_n = 1$, $n > 0$. How is it that uniqueness of representation can be claimed when clearly, for example, $[2; 3, 4, 2]$ and $[2; 3, 4, 1, 1]$ are the same number? The answer is that $a_n = 1$, $n > 0$

requires that $r_{n-2} = r_{n-1} = 1$, which violates the "uniqueness" assumption used in fixing a_i and r_i in the proof above—specifically note that the condition $0 < r_{n-1} < r_{n-2}$ in (5.53) is violated. If one is allowed to violate the required ever-decreasing remainders, then a_i and r_i cannot be uniquely fixed at each step of the proof, above.

An *intermediate fraction* is a fraction represented by the continued fraction representation of a k-th order convergent with the final partial quotient a_k reduced (this can naturally only be done when $a_k > 1$). As Khinchin points out ([59], p. 14): "*In arithmetic applications, these intermediate fractions play an important role (though not as important a role as the convergents)*".

The intermediate fractions (of a k-th order convergent) form a monotonically increasing or decreasing sequence of fractions ([59], p. 13):

$$\frac{p_{k-2}}{q_{k-2}}, \frac{p_{k-2} + p_{k-1}}{q_{k-2} + q_{k-1}}, \frac{p_{k-2} + 2p_{k-1}}{q_{k-2} + 2q_{k-1}}, \ldots, \frac{p_{k-2} + a_k p_{k-1}}{q_{k-2} + a_k q_{k-1}} = \frac{p_k}{q_k}. \qquad (5.49)$$

Note in (5.49) that the first and last fractions are not intermediate fractions (rather, they are convergents).

5.6 Euclid's GCD Algorithm

The apparatus of continued fractions is closely related to Euclid's GCD algorithm (in fact, historically, Euclid's GCD algorithm is considered a precursor of continued fractions). It was noted in Lemma 5.13 that the last non-zero remainder when Algorithm 5.1 is applied is the greatest common divisor of a and b. In this section, we present Euclid's algorithm, prove it, and show it similarity to Algorithm 5.1.

Knuth ([3], p. 335) presents some background information about Euclid's GCD algorithm:

> Euclid's algorithm is found in Book 7, Propositions 1 and 2 of his *Elements* (c. 300 B.C.), but it probably wasn't his own invention. Some scholars believe that the method was known up to 200 years earlier, at least in its subtractive form, and it was almost certainly known to Eudoxus (c. 375 B.C.); see K. von Fritz, *Ann. Math.* (2) **46** 1945, 242-264. Aristotle (c. 330 B.C.) hinted at it in his *Topics*, 158b, 29-35. However, very little hard evidence about such early history has survived [see. W. R. Knorr, *The Evolution of the Euclidian Elements* (Dordrecht: 1975)].

> We might call Euclid's method the granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day. (The chief rival for this honor is perhaps the ancient Egyptian method for multiplication, which was based on doubling and adding, and which forms the basis for efficient calculation of nth powers as explained in section 4.6.3. ...)

Algorithm 5.15 (Euclid's GCD Algorithm For Greatest Common Divisor Of Positive Integers a And b):[18]

- If $(a < b)$, swap a and b.[19]

- Repeat

  - $r := a \bmod b$.
  - If $(r = 0)$, STOP.
  - $a := b$.
  - $b := r$.

- **Exit condition:** b will be the g.c.d. of a and b.

Proof: Olds ([61], pp. 16-17) shows the relationship between Algorithm 5.1 and Euclid's algorithm, and presents a proof, which is reproduced nearly verbatim here.

First, note that d is the GCD of a and b iff:

- (Necessary Condition I) d divides both a and b, and

- (Necessary Condition II) any common divisor c of a and b divides d.

Essentially, we will prove that the final non-zero remainder when the algorithm is applied meets the two criteria above, and hence must be the GCD of a and b.

Note that (5.6) through (5.10) can be rewritten as (5.50) through (5.54), which make them consistent with the form Olds' presents.

$$a = a_0 b + r_0, \qquad 0 < r_0 < b \tag{5.50}$$
$$b = a_1 r_0 + r_1, \qquad 0 < r_1 < r_0 \tag{5.51}$$
$$r_0 = a_2 r_1 + r_2, \qquad 0 < r_2 < r_1 \tag{5.52}$$
$$\cdots$$
$$r_{n-3} = a_{n-1} r_{n-2} + r_{n-1}, \qquad 0 < r_{n-1} < r_{n-2} \tag{5.53}$$
$$r_{n-2} = a_n r_{n-1} + 0, \qquad 0 = r_n \tag{5.54}$$

First, we will show that *Necessary Condition I*, above, is met. Note from (5.54) that $r_{n-1} \mid r_{n-2}$, since r_{n-2} is an integer multiple of r_{n-1}. Note from (5.53) that $r_{n-1} \mid r_{n-3}$, since r_{n-3} is also an integer multiple of r_{n-1}. This logic can be

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[18]Knuth ([3], pp. 336-337) distinguishes between the *original* Euclidian algorithm and the *modern* Euclidian algorithm. The algorithm presented here is more closely patterned after the *modern* Euclidian algorithm.

[19]This step isn't strictly necessary, but is usually done to save one iteration.

Table 5.2: Euclid's Algorithm Applied To Find Greatest Common Divisor Of 1,736,651 and 26,023 (Example 5.16)

Iteration	a	b	$r := a \bmod b$
1	1,736,651	26,023	19,133
2	26,023	19,133	6,890
3	19,133	6,890	5,353
4	6,890	5,353	1,537
5	5,353	1,537	742
6	1,537	742	53
7	742	53	0

carried "upward" in the set of equations represented by (5.50) through (5.54), and we can finally conclude that $r_{n-1} \mid b$ and $r_{n-1} \mid a$. This proves *Necessary Condition I*.

Second, we will show that *Necessary Condition II*, above, is met. This time, in (5.50) through (5.54), we work top-down rather than bottom-up. Assume that c is a divisor of a and a divisor of b. Then, from the form of (5.50), c divides r_0.[20] Similarly, from the form of (5.51), c divides r_1. This rationale can be carried "downward" to finally conclude that c divides r_{n-1}. Thus, $(c \mid a) \wedge (c \mid b) \rightarrow (c \mid r_{n-1})$, where r_{n-1} is the last non-zero remainder. This proves *Necessary Condition II*.

Thus, r_{n-1} is the GCD of a and b. ∎

Remarks: It is easy to observe that the only difference between Algorithm 5.1 and Algorithm 5.15 is that Algorithm 5.1 records the quotient of each division, whereas Algorithm 5.15 does not.

Example 5.16: Use Euclid's algorithm to find the greatest common divisor of 1,736,651 and 26,023.

Solution: Table 5.2 shows the application of Algorithm 5.15 (Euclid's GCD algorithm) to find the greatest common divisor of 1,736,651 and 26,023. The last non-zero remainder (and hence the greatest common divisor) is 53.

Remarks: The prime factorization of 1,736,651 is $151 \times 53 \times 31 \times 7$, and the prime factorization of 26,023 is 491×53, which is consistent with the result in Table 5.2.

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[20]This implication may be counterintuitive at first glance. It concerns "reachability" of linear combinations of integers with a common divisor. Specifically, if a and b have a common divisor c, any linear combination $ia + jb$, $(i, j \in \mathbb{Z})$, can "reach" only multiples of c. In (5.50), $(1)(a) + (-a_0)(b) = r_0$, thus r_0 must be a multiple of c. An identical argument applies for (5.50) through (5.54).

5.7   Continued Fraction Representation Of Irrational Numbers

Irrational numbers (such as $\sqrt{2}$ or π) necessarily have infinite continued fraction representations (i.e. the representations do not terminate). This is clear, since Theorem 5.6 gives a concrete procedure for constructing a rational number from any *finite* continued fraction; therefore a continued fraction corresponding to an irrational number cannot be finite.

The algorithm for determining the partial quotients of an irrational number is awkward, because it is a symbolic (rather than a numerical) algorithm. We present the algorithm here for perspective and completeness, although it is not often useful in practical engineering work. In practical work, an ordinary hand-held calculator will supply a real number to far more precision than necessary, and the displayed real number can be converted to a rational number for the application of Algorithm 5.1. For practical work, it is rarely necessary to apply a Algorithm 5.17.

The symbolic algorithm for determining the continued fraction partial quotients of a real number is a recursive process very similar to the algorithm for determining the continued fraction partial quotients of a rational number. The essential activity is choosing the largest possible integer a_i in each iteration.

Algorithm 5.17 begins by choosing the largest integer a_0 not larger than x, then expressing x as

$$x = a_0 + \frac{1}{x_1}. \tag{5.55}$$

With a_0 chosen, x_1 can then be expressed as

$$x_1 = \frac{1}{x - a_0}. \tag{5.56}$$

x_1 can then be expressed as

$$x_1 = a_1 + \frac{1}{x_2}, \tag{5.57}$$

and a_1, the largest integer not larger than x_1, can be chosen. This process can be continued indefinitely (with an irrational x, it won't terminate) to determine as many partial quotients as desired.

Algorithm 5.17 (Symbolic Algorithm For Obtaining Continued Fraction Representation Of An Irrational Number x):

- $x_0 := x$ (the real number whose partial quotients are desired).

- $k := -1$.

- Repeat

  - $k := k + 1$.
  - $a_k := \lfloor x_k \rfloor$.
  - $x_{k+1} := \dfrac{1}{x_k - a_k}$.

- Until (as many partial quotients as desired are obtained).

Example 5.18: Find the first several continued fraction partial quotients of $\sqrt{3}$.

Solution: Applying Algorithm 5.17:

$$x_0 := x = \sqrt{3} \tag{5.58}$$

$$k := -1 \tag{5.59}$$

$$k := k + 1 = 0 \tag{5.60}$$

$$a_0 := \lfloor x_0 \rfloor = \lfloor \sqrt{3} \rfloor = 1 \tag{5.61}$$

$$x_1 := \frac{1}{x_0 - a_0} = \frac{1}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{2} \tag{5.62}$$

$$k := k + 1 = 1 \tag{5.63}$$

$$a_1 := \lfloor x_1 \rfloor = \left\lfloor \frac{\sqrt{3} + 1}{2} \right\rfloor = 1 \tag{5.64}$$

$$x_2 := \frac{1}{x_1 - 1} = \frac{1}{\frac{\sqrt{3}+1}{2} - 1} = \sqrt{3} + 1 \tag{5.65}$$

$$k := k + 1 = 2 \tag{5.66}$$

$$a_2 := \lfloor x_2 \rfloor = \lfloor \sqrt{3} + 1 \rfloor = 2 \tag{5.67}$$

$$x_3 := \frac{1}{(\sqrt{3} + 1) - 2} = \frac{\sqrt{3} + 1}{2} \tag{5.68}$$

Note that $x_3 = x_1$, so the algorithm will repeat with $a_3 = 1$, $a_4 = 2$, $a_5 = 1$, $a_6 = 2$, etc. Thus, the continued fraction representation of $\sqrt{3}$ is $[1; 1, 2, 1, 2, 1, 2, \ldots]$ $= [1; \overline{1, 2}]$.

Remarks: It can be proved that all continued fractions that repeat or repeat from some point onward represent real numbers of the form $\frac{P \pm \sqrt{D}}{Q}$, where $D \in \mathbb{N}$ is not the square of an integer. It can also be shown that all numbers of this form result in continued fractions that repeat or repeat from some point onward. (See Olds [61], Chapter 4.) It is beyond the scope of our interest in continued fractions to develop these properties.

5.8   Convergents As Best Approximations

Up until this point, we've presented general properties of continued fractions and convergents without regard for practical applications. In this section, we present results and algorithms to use the apparatus of continued fractions to obtain best rational approximations.

Although we don't dwell on other algorithms for locating best rational approximations (we present only the single best algorithm), it is worth noting that there are many naive algorithms for locating best rational approximations. These include:

- Exhaustive search of the integer lattice $[O(h_{MAX} k_{MAX})]$.

- Building the Farey series starting at an integer $[O(max(h_{MAX}, k_{MAX})^2)]$ (see Algorithm 4.12).

- Building the Farey series starting at a rational number with a large prime denominator $[O(max(h_{MAX}, k_{MAX}))]$.

- Building the Stern-Brocot tree (see Section 5.9) $[O(max(h_{MAX}, k_{MAX}))]$.

Although we don't justify it formally, the continued fraction algorithms presented here are $O(log\ max(h_{MAX}, k_{MAX}))$.[21] The basis on which we make that assertion is the geometric rate of increase of convergents (see Theorem 5.12), which means that the number of steps required is tied to the logarithm of the maximum denominator involved, as it is necessary to obtain partial quotients and convergents only until $q_k \geq max(h_{MAX}, k_{MAX})$.

Theorem 5.19: In the case of an infinite continued fraction for $k \geq 0$ or in the case of a finite continued fraction for $0 \leq k < n-1$, a convergent $s_k = p_k/q_k$ to a [rational or irrational] number $\alpha \in \mathbb{R}^+$ satisfies

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}. \tag{5.69}$$

In the case of a finite continued fraction with $k = n - 1$,

$$\left| \alpha - \frac{p_k}{q_k} \right| = \frac{1}{q_k q_{k+1}}. \tag{5.70}$$

Proof: The proof comes directly from Theorem 5.7 (Corollary I) and Theorem 5.11. ∎

Remarks: Khinchin describes this result ([59], p. 9) as playing a basic role in the arithmetic applications of continued fractions. In fact, this theorem is used in the proof of Theorem 5.20.

We now present and prove the fundamental theorem of this chapter, which gives an $O(log\ N)$ algorithm for finding the enclosing neighbors in F_N to an arbitrary rational number a/b.[22]

Theorem 5.20 (Enclosing Neighbors Of $x \notin F_N$ In F_N): For a non-

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[21] Well, not exactly. In the classical computer science sense (speaking only in terms of number of operations), the algorithms are $O(log\ max(h_{MAX}, k_{MAX}))$. However, if h_{MAX} and k_{MAX} are increased beyond the sizes of integers that a computer can inherently accomodate, one must use long integer calculation software, which requires more time for each integer operation than is required for machine native integer sizes. As h_{MAX} and k_{MAX} are increased far beyond integer sizes accomodated inherently by the computer, the relationship surely is not $O(log\ max(h_{MAX}, k_{MAX}))$.

[22] Theorem 5.20 applies to irrational numbers as well, so long as one can obtain enough partial quotients, but we don't highlight this fact because it is rare in engineering applications that one uses symbolic methods to obtain best rational approximations. As emphasized by (5.3), (5.4), and (5.5), the process of obtaining partial quotients is essentially a process of determining in which partition a number lies. All numbers in the same partition—rational or irrational—have the same Farey neighbors in all Farey series up to a certain order. If the partial quotients of an irrational number can be obtained up through a_k s.t. $s_k = p_k/q_k$ is the highest-order convergent with $q_k \leq N$, then this theorem can be applied. Knowledge of all $a_0, \ldots, a_k$ is equivalent to the knowledge that the number is in a partition where all numbers in that partition have the same Farey neighbors in all Farey series up through at least order q_k.

negative rational number a/b not in F_N which has a continued fraction representation $[a_0; a_1, a_2, \ldots, a_n]$, the highest-order convergent $s_k = p_k/q_k$ with $q_k \leq N$ is one neighbor[23] to a/b in F_N, and the other neighbor in F_N is[24]

$$\frac{\left\lfloor \dfrac{N - q_{k-1}}{q_k} \right\rfloor p_k + p_{k-1}}{\left\lfloor \dfrac{N - q_{k-1}}{q_k} \right\rfloor q_k + q_{k-1}}. \tag{5.71}$$

Proof: First, it is proved that the highest-order convergent $s_k = p_k/q_k$ with $q_k \leq N$ is one of the two neighbors to a/b in F_N. $s_k \in F_N$, since $q_k \leq N$. By Theorem 5.19, the upper bound on the difference between a/b and arbitrary s_k is given by

$$\left| \frac{a}{b} - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}. \tag{5.72}$$

For two consecutive terms in F_N, $Kh - Hk = 1$ (Theorem 4.8). For a Farey neighbor H/K to s_k in F_N, (5.73) must hold.

$$\frac{1}{q_k N} \leq \left| \frac{H}{K} - \frac{p_k}{q_k} \right| \tag{5.73}$$

$q_{k+1} > N$, because $q_{k+1} > q_k$ and p_k/q_k was chosen to be the highest-order convergent with $q_k \leq N$. Using this knowledge and combining (5.72) and (5.73) leads to (5.74).

$$\left| \frac{a}{b} - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}} < \frac{1}{q_k N} \leq \left| \frac{H}{K} - \frac{p_k}{q_k} \right| \tag{5.74}$$

This proves that s_k is one neighbor to a/b in F_N. The apparatus of continued fractions ensures that the highest order convergent s_k with $q_k \leq N$ is closer to a/b than to any neighboring term in F_N. Thus, there is no intervening term of F_N between s_k and a/b. If k is even, $s_k < a/b$, and if k is odd, $s_k > a/b$.

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[23]By neighbors in F_N we mean the rational numbers in F_N immediately to the left and immediately to the right of a/b.

[24]Theorem 5.20 is a somewhat stronger statement about best approximations than Khinchin makes in [59], Theorem 15.   We were not able to locate this theorem or a proof in print, but this theorem is understood within the number theory community.   It appears on the Web page of David Eppstein [24] in the form of a 'C'-language computer program, `http://www.ics.uci.edu/~eppstein/numth/frap.c`. Although Dr. Eppstein phrases the solution in terms of modifying a partial quotient, his approach is equivalent to (5.71).

It must be proved that (5.71) is the other Farey neighbor. For brevity, only the case of k even is proved: the case of k odd is symmetrical. (5.71) is of the form (5.75), where $i \in \mathbb{Z}^+$.

$$\frac{ip_k + p_{k-1}}{iq_k + q_{k-1}} \tag{5.75}$$

k is even, $s_k < a/b$, and the two Farey terms enclosing a/b, in order, are

$$\frac{p_k}{q_k}, \frac{ip_k + p_{k-1}}{iq_k + q_{k-1}}. \tag{5.76}$$

Applying the $Kh - Hk = 1$ test, (5.77), gives the result of 1, since by Theorem 5.7, $q_k p_{k-1} - p_k q_{k-1} = (-1)^k$.

$$(q_k)(ip_k + p_{k-1}) - (p_k)(iq_k + q_{k-1}) = 1 \tag{5.77}$$

Thus, every potential Farey neighbor of the form (5.75) meets the $Kh - Hk = 1$ test. It is also straightforward to show that *only* potential Farey neighbors of the form (5.75) can meet the $Kh - Hk = 1$ test, using the property that p_k and q_k are coprime.

It must be established that a rational number of the form (5.75) is irreducible. This result comes directly from (5.77), since if the numerator and denominator of (5.71) or (5.75) are not coprime, the difference of 1 is not possible.

The denominator of (5.71) can be rewritten as

$$N - [(N - q_{k-1}) \bmod q_k] \in \{N - q_k + 1, ..., N\}. \tag{5.78}$$

It must be shown that if one irreducible rational number—namely, the rational number given by (5.71)—with a denominator $\in \{N - q_k + 1, \ldots, N\}$ meets the $Kh - Hk = 1$ test, there can be no other irreducible rational number in F_N with a larger denominator which also meets this test.

Given (5.77), and given that *only* rational numbers of the form (5.75) can meet the $Kh - Hk = 1$ test, and given that any number of the form (5.75) is irreducible, the irreducible number meeting the $Kh - Hk = 1$ test with the next larger denominator after the denominator of (5.71) will have a denominator $\in \{N + 1, \ldots, N + q_k\}$. Thus, no other irreducible rational number in F_N besides that given by (5.71) with a larger denominator $\leq N$ and which meets the $Kh - Hk = 1$ test can exist; therefore (5.71) is the other enclosing Farey neighbor to a/b in F_N. ∎

Theorem 5.20 establishes that the two neighbors in F_N to a rational number a/b will be the highest-order convergent with a denominator not exceeding N,

and the number specified by (5.71). An interesting and worthwhile question to ask about Theorem 5.20 is which of the two neighbors will be *closer* to a/b—the convergent or the number specified by (5.71)? Can we make any strong, simple, and easy-to-remember statements about the relative distance? We answer this question and some related questions now.

We are not aware of any rules that decisively predict which of the two Farey neighbors in Theorem 5.20 will be closer to a/b[25], although Lemma 5.25 is able to predict that the highest-ordered convergent s_k with a denominator not exceeding N will be closer in many cases. In general, either neighbor may be closer. The most straightforward approach that we are aware of is to calculate both Farey neighbors and to calculate their respective distances from a/b. The difficulty in devising a simple rule to predict which neighbor is closer is compounded by that fact that knowledge of $[a_0; a_1, \ldots, a_k]$ such that s_k is the highest-ordered convergent with $q_k \leq N$ is incomplete knowledge of a/b and can only confine a/b to an inequality in the sense suggested by (5.3) through (5.5). Note that the value specified by (5.71) is an intermediate fraction, and that the statement of Theorem 5.20 coincides with Khinchin's Theorem 15 ([59], p. 22).

However, even in the absence of a rule to decisively predict which of the two Farey neighbors specified by Theorem 5.20 is closer to a/b, there are other useful properties of convergents as best approximations which we present now.

It has been stated earlier that even-ordered convergents form an increasing sequence and that odd-ordered convergents also form a decreasing sequence. However, up to this point, we have not made a statement about the relationship between even- and odd-ordered convergents. We present such a statement as Lemma 5.21, below.

Lemma 5.21:   In the case of a finite or infinite continued fraction representation of a non-negative rational or irrational number $\alpha \in \mathbb{R}^+$, for all k,

$$\left| \alpha - \frac{p_k}{q_k} \right| < \left| \alpha - \frac{p_{k-1}}{q_{k-1}} \right|.$$

(5.79)

In other words, convergents get ever-closer to α, without respect to whether they are even- or odd-ordered convergents.
Proof: In this proof, we show that for all k,

$$|s_{k-2} - s_{k-1}| > 2|s_{k-1} - s_k|.$$

(5.80)

To understand why the proof is valid, consider the case of k even, in which case $s_k < \alpha$, so that $s_{k-1} - \alpha < s_{k-1} - s_k$. If $s_{k-1} - s_{k-2} > 2(s_{k-1} - s_k)$, then s_{k-2}

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[25]We should qualify this by saying that we mean a rule which uses only partial quotients up through a_k or at most a_{k+1}, which is the same information used by the theorem. We should also add that although Theorem 5.20 is worded to only consider non-negative rational numbers, the theorem and the results here apply to non-negative irrational numbers as well, so long as enough partial quotients can be obtained.

is further to the left of α than s_{k-1} is to the right of α; thus (5.79) applies. A symmetrical argument holds for k odd.

By Theorem 5.7,

$$s_{k-2} - s_{k-1} = \frac{p_{k-2}}{q_{k-2}} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{k-1}}{q_{k-2}q_{k-1}}, \tag{5.81}$$

and similarly

$$s_{k-1} - s_k = \frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_{k-1}q_k}. \tag{5.82}$$

In order for (5.80) to be met, it must be true that $2q_{k-2}q_{k-1} < q_{k-1}q_k$, or equivalently that $2q_{k-2} < q_k$. Since canonically $q_k = a_k q_{k-1} + q_{k-2}$ (Eq. 5.29), the requirement is that $2q_{k-2} < a_k q_{k-1} + q_{k-2}$. Since $a_k \geq 1$ and convergents are ever-increasing, (5.80) is met and the lemma is proved. ∎

Theorem 5.19 establishes a maximum distance from a number α that we wish to approximate to a convergent. We now provide a second result that establishes a *minimum* distance. (This result is Theorem 13, p. 15, from [59].)

Theorem 5.22: In the case of an infinite continued fraction representation $[a_0; a_1, a_2, \ldots]$ of a non-negative irrational number $\alpha \in \mathbb{R}^+$, for all $k \geq 0$; or in the case of a [necessarily finite] continued fraction representation $[a_0; a_1, a_2, \ldots, a_n]$ of a non-negative rational number $\alpha \in \mathbb{R}^+$, for all $0 \leq k \leq n-1$,

$$\left| \alpha - \frac{p_k}{q_k} \right| > \frac{1}{q_k(q_{k+1} + q_k)}. \tag{5.83}$$

Proof: We've already established (Lemma 5.21) that each convergent s_{k+1} is nearer to a number α to be approximated than the previous convergent, s_k, i.e. for all k,

$$\left| \alpha - \frac{p_{k+1}}{q_{k+1}} \right| < \left| \alpha - \frac{p_k}{q_k} \right|. \tag{5.84}$$

Since the mediant of two fractions always lies between them (Lemma 4.5), it follows directly that

$$\left| \alpha - \frac{p_k}{q_k} \right| > \left| \frac{p_k + p_{k+1}}{q_k + q_{k+1}} - \frac{p_k}{q_k} \right| = \frac{1}{q_k(q_k + q_{k+1})}. \tag{5.85}$$

■

Remark I: This theorem can be combined with Theorem 5.19 to give the following combined inequality:

$$\frac{1}{q_k(q_k + q_{k+1})} < \left| \alpha - \frac{p_k}{q_k} \right| \leq \frac{1}{q_k q_{k+1}}. \tag{5.86}$$

We now supply an interesting and sometimes useful property of convergents used as best approximations. Note that we later show that Lemma 5.23 is a weak statement (a stronger statement can be made, Lemma 5.24), but this lemma has the advantage of being extremely easy to remember.

Lemma 5.23: A convergent $s_k = p_k/q_k$ to a non-negative [rational or irrational] number $\alpha \in \mathbb{R}^+$ is closer to α than any other rational number with the same or a smaller denominator.
Proof: Let α be the non-negative real number, rational or irrational, that we wish to approximate.

If there is a number (let's call it c/d) closer to α than $s_k = p_k/q_k$, with the same or a smaller denominator than s_k, then by definition it must be in the Farey series of order q_k, which we denote F_{q_k}.

Theorem 5.20 assures us that the two Farey neighbors to α in F_{q_k} will be s_k and the number given by (5.71). Note that Theorem 5.20 applies to irrational numbers as well (although the theorem statement does not indicate this), so we interpret Theorem 5.20 in that sense.

Note in (5.71) that the expression involving the $floor(\cdot)$ function will evaluate to be zero, since $N = q_k$. Thus, the other Farey neighbor to α in F_{q_k} will be $s_{k-1} = p_{k-1}/q_{k-1}$.

We have already shown in Lemma 5.21 that $|\alpha - s_{k-1}| > |\alpha - s_k|$, therefore s_k is closer to α than the other Farey neighbor given by (5.71). Furthermore, because any c/d which is closer to α than s_k must be present in F_{q_k}, such a c/d does not exist.   ■
Remark I: In practice, this lemma is little more than parlor trivia (it is not mathematically significant), but it is useful information and very easy to remember. For example, $355/113$ is a convergent to π, and it is sometimes useful to know that no better rational approximation can exist with the same or a smaller denominator.
Remark II: A stronger statement can be made (see Lemma 5.24).

We now present a stronger statement about convergents as best approximations that is not as easy to remember as Lemma 5.23.

Lemma 5.24: A convergent $s_k = p_k/q_k$ to a non-negative [rational or irrational] number $\alpha \in \mathbb{R}^+$ is closer to α than any other rational number with a

denominator less than $q_k + q_{k-1}$.

Proof: Let N be the denominator of a rational number which is potentially closer to α than s_k. If $N < q_k + q_{k+1}$, then (5.71) evaluates to s_{k-1}, and Lemma 5.21 has established that s_k is closer to α than s_{k-1}. If, on the other hand, $N \geq q_k + q_{k+1}$, then the intermediate fraction specified by (5.71) *may* be closer to α than s_k. ∎

Remark I: This statement is harder to remember, but a stronger statement than Lemma 5.23.

Remark II: Note that the only valid implication is $N < q_k + q_{k+1} \rightarrow$ (convergent is closer). Note that $N \geq q_k + q_{k+1} \not\rightarrow$ (intermediate fraction is closer)! If $N \geq q_k + q_{k+1}$, either the convergent or the intermediate fraction may be closer. This statement is harder to remember, but a stronger statement than Lemma 5.23.

Finally, we present a result about Theorem 5.20 that will predict in some circumstances that the highest-ordered convergent s_k with a denominator not exceeding N must be closer to a/b than the intermediate fraction specified by (5.71).

Lemma 5.25: In Theorem 5.20, if $N < q_k + q_{k-1}$, the highest ordered convergent s_k with a denominator not exceeding N is closer to a/b[26] then the intermediate fraction specified by (5.71). If $N \geq q_k + q_{k-1}$, either s_k or the intermediate fraction specified by (5.71) may be closer.

Proof: See the proof of Lemma 5.24. ∎

Theorem 5.20 immediately suggests an algorithm for obtaining the enclosing rational numbers in F_N to a rational number $a/b \notin F_N$, which we present as Algorithm 5.26. Although we don't formally show it, the algorithm is $O(log\ N)$, due to the minimum geometric rate of increase of convergents (Theorem 5.12). Note that the algorithm will proceed only until $q_k > N$, not necessarily until all partial quotients of a/b are obtained. Note also that the algorithm can be applied to irrational numbers with minor modification (all that matters is that we can obtain enough partial quotients).

Algorithm 5.26 (Enclosing Neighbors Of $a/b \notin F_N$ In F_N):

- $k := -1$.

- $divisor_{-1} := a$.

- $remainder_{-1} := b$.

- $p_{-1} := 1$.

- $q_{-1} := 0$.

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[26] Note that this result is also valid for convergents to an irrational number, although Theorem 5.20 is not worded in this way.

Table 5.3: Application Of Algorithm 5.26 To Find Members Of F_{100} Which Enclose 10,000/16,093 (Example 5.27)

Index (k)	$dividend_k$	$divisor_k$	a_k	$remainder_k$	p_k	q_k
-1	N/A	10,000	N/A	16,093	1	0
0	10,000	16,093	0	10,000	0	1
1	16,093	10,000	1	6,093	1	1
2	10,000	6,093	1	3,907	1	2
3	6,093	3,907	1	2,186	2	3
4	3,907	2,186	1	1,721	3	5
5	2,186	1,721	1	465	5	8
6	1,721	465	3	326	18	29
7	465	326	1	139	23	37
8	326	139	2	48	64	103

- Repeat

    - $k := k + 1$.
    - $dividend_k := divisor_{k-1}$.
    - $divisor_k := remainder_{k-1}$.
    - $a_k := dividend_k \ div \ divisor_k$.
    - $remainder_k := dividend_k \ mod \ divisor_k$.
    - If $k = 0$ then $p_k := a_k$ else $p_k := a_k p_{k-1} + p_{k-2}$.
    - If $k = 0$ then $q_k := 1$ else $q_k := a_k q_{k-1} + q_{k-2}$.

- Until $(q_k > k_{MAX})$.

- $s_{k-1} = p_{k-1}/q_{k-1}$ will be one Farey neighbor to a/b in $F_{k_{MAX}}$. Apply (5.71) to obtain the other Farey neighbor.

Example 5.27:   Find the members of F_{100} which enclose the conversion factor from kilometers-per-hour to miles-per-hour. Assume that one mile is 1.6093 kilometers (exactly).
Solution: The conversion factor from KPH to MPH is the reciprocal of 1.6093. As a rational number, 1.6093 is 16,093/10,000, so 10,000/16,093 is its exact reciprocal. Applying Algorithm 5.26 with $a/b = 10,000/16,093$ and $k_{MAX} = 100$ yields Table 5.3.

Note from Table 5.3 that the 7-th order convergent, $s_7 = 23/37$, is the highest-ordered convergent with $q_k \leq 100$, so by Theorem 5.20, 23/37 is one neighbor in F_{100} to 10,000/16,093. Because s_7 is an odd-ordered convergent, it will be

Table 5.4: Application Of Algorithm 5.26 To Find Members Of F_{100} Which Enclose $\sqrt{3}$ (Example 5.28)

Index (k)	a_k	p_k	q_k
-1	N/A	1	0
0	1	1	1
1	1	2	1
2	2	5	3
3	1	7	4
4	2	19	11
5	1	26	15
6	2	71	41
7	1	97	56
8	2	265	153
9	1	362	209

the right Farey neighbor. By (5.71), the other Farey neighbor is 41/66, and it will be the left Farey neighbor.

Example 5.28:   Find the members of F_{200} which enclose $\sqrt{3}$.
Solution: We demonstrated in Example 5.18 that the continued fraction representation of $\sqrt{3}$ is $[1; \overline{1,2}]$. As is highlighted in Footnote 22, it isn't required that a number be rational to apply Theorem 5.20, so long as enough partial quotients can be obtained. Using knowledge of the partial quotients of $\sqrt{3}$ and applying Algorithm 5.26 yields Table 5.4 (note that it isn't necessary to track remainders, as we already have all of the partial quotients for $\sqrt{3}$).

Note from Table 5.4 that the 8-th order convergent, $s_8 = 265/153$, is the highest-ordered convergent with $q_k \leq 200$, so by Theorem 5.20, 265/153 is one neighbor in F_{100} to $\sqrt{3}$. Because s_8 is an even-ordered convergent, it will be the left Farey neighbor. By (5.71), the other Farey neighbor is 97/56, and it will be the right Farey neighbor.

It is clear that Algorithm 5.26 can be trivially modified to find enclosing neighbors in $F_{k_{MAX}, \overline{h_{MAX}}}$, and we present this trivial modification as Algorithm 5.29.

Algorithm 5.29 (Enclosing Neighbors Of $x \notin F_{k_{MAX}, \overline{h_{MAX}}}$ In $F_{k_{MAX}, \overline{h_{MAX}}}$):

- If $a/b < h_{MAX}/k_{MAX}$, apply Algorithm 5.26 directly;

- Else if $a/b > h_{MAX}/k_{MAX}$, apply Algorithm 5.26 using b/a rather than a/b as the input to the algorithm, using h_{MAX} rather than k_{MAX} as N, and

Table 5.5: Application Of Algorithm 5.29 To Find Members Of F_{100} Which Enclose $1/\sqrt{3}$ (Example 5.30)

Index (k)	a_k	p_k	q_k
-1	N/A	1	0
0	0	0	1
1	1	1	1
2	1	1	2
3	2	3	5
4	1	4	7
5	2	11	19
6	1	15	26
7	2	41	71
8	1	56	97
9	2	153	265

using the reciprocals of the results of the algorithm.[27]

Example 5.30:   Find the members of $F_{200,\overline{100}}$ which enclose $\sqrt{3}$.
Solution: It was shown in Example 5.28 that the two enclosing neighbors to $\sqrt{3}$ in F_{200} are $265/153$ and $97/56$. Note that the first of these neighbors, $265/153$, violates the constraint on the numerator. As explained in Section 4.8, because $\sqrt{3} > 100/200$, the constraint on the numerator is the dominant constraint, and the necessary approach is to find the neighbors of $1/\sqrt{3}$ in F_{100}, then invert the results. Although we don't explain it in this work, the reciprocal of a continued fraction can be formed by "right-shifting" or "left-shifting" the continued fraction one position. Thus, if $[1; 1, 2, 1, 2, 1, 2, \ldots] = [1; \overline{1, 2}]$ is the continued fraction representation of $\sqrt{3}$, then $[0; 1, 1, 2, 1, 2, 1, \ldots] = [0; 1, \overline{1, 2}]$ is the continued fraction representation of $1/\sqrt{3}$. Using this result and constructing the convergents until $q_k \geq 100$ yields Table 5.5.

Note from Table 5.5 that the 8-th order convergent, $s_8 = 56/97$, is the highest-ordered convergent with $q_k \leq 100$, so by Theorem 5.20, $56/97$ is one neighbor in F_{100} to $1/\sqrt{3}$. Because s_8 is an even-ordered convergent, it will be the left Farey neighbor. By (5.71), the other Farey neighbor is $41/71$, and it will be the right Farey neighbor. Taking the reciprocal of these neighbors (and reversing their order) yields $97/56 < \sqrt{3} < 41/71$ as the two members of $F_{200,\overline{100}}$ which enclose $\sqrt{3}$.

A natural question to ask is whether, given only a *single* rational number $a/b \in F_N$, the apparatus of continued fractions can be used to economically find

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[27]The basis for taking the reciprocals of input and output and using h_{MAX} rather than k_{MAX} are explained in Section 4.8.

Table 5.6: Partial Quotients And Convergents Of 5/7 (Example 5.32)

Index (k)	$dividend_k$	$divisor_k$	a_k	$remainder_k$	p_k	q_k
-1	N/A	5	N/A	7	1	0
0	5	7	0	5	0	1
1	7	5	1	2	1	1
2	5	2	2	1	2	3
3	2	1	2	0	5	7

its neighbors in F_N. Examining the proof of Theorem 5.20, we see that the entire proof applies even if the denominator of the highest-order convergent, q_n, is less than or equal to N—that is, the number specified by (5.71) is a left or right Farey neighbor in F_N of a/b. If n is even, $s_{n-1} > s_n$, and the number specified by (5.71) will be the right Farey neighbor of s_n, and (4.34) and (4.35) can be used to find the left Farey neighbor. On the other hand if n odd, $s_{n-1} < s_n$, (5.71) will be the left Farey neighbor of s_n, and (4.32) and (4.33) can be used to find the right Farey neighbor. We summarize these observations as Algorithm 5.31.

Algorithm 5.31 (Neighbors Of $a/b \in F_N$ In F_N):

- Apply the first part of Algorithm 5.26 to obtain all of the partial quotients and convergents of a/b. The final convergent, $s_n = p_n/q_n$, will be a/b in reduced form.

- Use (5.71) (with $k = n$) to obtain the right Farey neighbor (if n is even) or the left Farey neighbor (if n is odd).

- If n is even, $s_{n-1} > s_n$, and the number specified by (5.71) will be the right Farey neighbor of s_n. Use (4.34) and (4.35) to find the left Farey neighbor. On the other hand if n is odd, $s_{n-1} < s_n$, (5.71) will be the left Farey neighbor of s_n. Use (4.32) and (4.33) to find the right Farey neighbor.

Example 5.32: Find the neighbors of 5/7 in $F_{1,000,000}$.
Solution: As per Algorithm 5.31, the first step is to obtain the partial quotients and convergents of 5/7 (these partial quotients and convergents are shown in Table 5.6).

Since the final convergent, s_3, is an odd-ordered convergent, $s_{k-1} < s_k$, and (5.71) will supply the left Farey neighbor of 5/7. Applying (5.71) with $N = 1,000,000$, $k = 3$, $k-1 = 2$, $p_k = 5$, $q_k = 7$, $p_{k-1} = 2$, and $q_{k-1} = 3$ yields $\frac{714,282}{999,995}$ as the left Farey neighbor of 5/7 in $F_{1,000,000}$. Application of (4.32) and

(4.33) yields $\frac{714,283}{999,996}$ as the right Farey neighbor.

Algorithm 5.33 (Neighbors Of $x \in F_{k_{MAX}, \overline{h_{MAX}}}$ In $F_{k_{MAX}, \overline{h_{MAX}}}$):

- If $a/b < h_{MAX}/k_{MAX}$, apply Algorithm 5.31 directly;

- Else if $a/b > h_{MAX}/k_{MAX}$, apply Algorithm 5.31 using b/a rather than a/b as the input to the algorithm, using h_{MAX} rather than k_{MAX} as N, and using the reciprocals of the results of the algorithm.[28]

5.9   The Stern-Brocot Tree

In this chapter, we've developed continued fraction techniques of best rational approximation without reference to any other models or theory. Because the algorithms presented in this chapter are $O(log\ N)$, the results so far are completely satisfactory and usable in practice. It is not necessary to go further or to present additional information.

However, there is a second model of best rational approximation (and a second set of algorithms), involving the Stern-Brocot tree. In fact, when reviewing the material in this chapter, some readers have inquired why the Stern-Brocot tree was not used.[29] In this section, we introduce the Stern-Brocot tree, demonstrate how to construct it, mention its major properties, show its correspondence with the apparatus of continued fractions, and finally show why we *must* use the apparatus of continued fractions to find best rational approximations.

5.9.1   Definition And Properties Of The Stern-Brocot Tree

The Stern-Brocot tree (Figure 5.1), is an infinite binary tree which contains all positive rational numbers.

To construct the tree, one begins with the two fractions $\frac{0}{1}$ and $\frac{1}{0}$, and forms the mediant (see Definition 4.4) of two adjacent fractions as many times as desired to generate additional fractions. Figure 5.1 illustrates the construction process. Note in Figure 5.1 that the adjacent fractions are always above and to the left and above and to the right of the fraction being constructed, and that in the construction of the Stern-Brocot tree, one of the adjacent fractions can be many levels upwards in the tree from the fraction being constructed. For example, in Figure 5.1, when constructing the fraction 4/5, the left adjacent fraction (3/4) is nearby in the figure, but the right adjacent fraction (1/1) is three levels up to the left and one level up to the right. Note when constructing the fraction 4/5 that its right adjacent fraction is *not* 4/3.

---

[28]The basis for taking the reciprocals of input and output and using h_{MAX} rather than k_{MAX} are explained in Section 4.8.

[29]In brief, the Stern-Brocot tree was not used because the resulting algorithms are $O(N)$, and so will introduce practical computational difficulties when used with large integers.
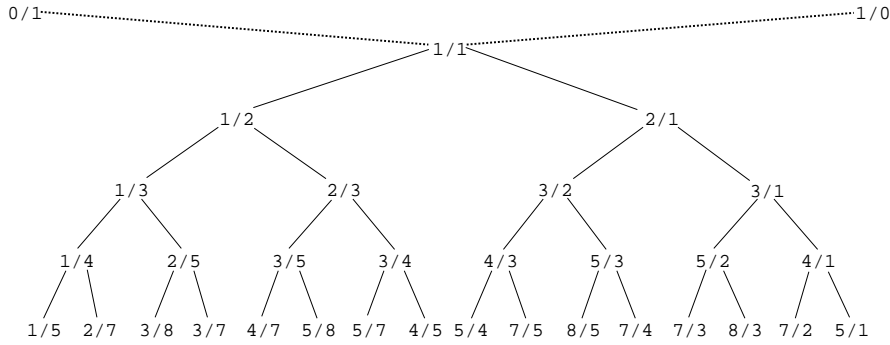
Figure 5.1: The Stern-Brocot Tree

Note that it is also possible to maintain the Stern-Brocot tree as an ordered list, rather than a tree, starting with the list $\{0/1, 1/0\}$. An additional element may be inserted between any two existing elements in the list by forming their mediant, and this process may be repeated indefinitely. Note also that two elements s_L and s_R are Farey neighbors to any number α if $s_L < \alpha < s_R$ and the mediant of s_L and s_R has a denominator larger than the order of the Farey series. This gives a convenient procedure for forming best rational approximations using only the Stern-Brocot tree, as the following example shows.

Example 5.34: Find the members of F_{10} which enclose π.
Solution: By repeatedly calculating mediants, terms can be added to the list $\{\frac{0}{1}, \frac{1}{0}\}$ until π is enclosed and it is not possible to generate additional enclosing terms whose denominator does not exceed 10. This process is carried out below.

$$\left\{\frac{0}{1}, \frac{1}{0}\right\}, \left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{0}\right\}, \left\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0}\right\}, \left\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}\right\}, \tag{5.87}$$

$$\left\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{1}{0}\right\}, \left\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{7}{2}, \frac{4}{1}, \frac{1}{0}\right\}, \left\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{10}{3}, \frac{7}{2}, \frac{4}{1}, \frac{1}{0}\right\}, \tag{5.88}$$

$$\left\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{13}{4}, \frac{10}{3}, \frac{7}{2}, \frac{4}{1}, \frac{1}{0}\right\}, \left\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{16}{5}, \frac{13}{4}, \frac{10}{3}, \frac{7}{2}, \frac{4}{1}, \frac{1}{0}\right\}, \tag{5.89}$$

$$\left\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{19}{6}, \frac{16}{5}, \frac{13}{4}, \frac{10}{3}, \frac{7}{2}, \frac{4}{1}, \frac{1}{0}\right\}, \tag{5.90}$$

$$\left\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{22}{7}, \frac{19}{6}, \frac{16}{5}, \frac{13}{4}, \frac{10}{3}, \frac{7}{2}, \frac{4}{1}, \frac{1}{0}\right\}, \tag{5.91}$$

$$\left\{ \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{25}{8}, \frac{22}{7}, \frac{19}{6}, \frac{16}{5}, \frac{13}{4}, \frac{10}{3}, \frac{7}{2}, \frac{4}{1}, \frac{1}{0} \right\}. \tag{5.92}$$

Note that 25/8 and 22/7 are the left and right neighbors to π in F_{10}, since $25/8 < \pi < 22/7$, and since the mediant of 25/8 and 22/7 (49/15) has a denominator which is too large for the Farey series being considered.

Note also that the construction process above could be trivially amended to treat the case of a constrained numerator rather than a constrained denominator.

The Stern-Brocot tree has many remarkable properties (especially in view of the simplicity of its construction). We mention the following properties without proof.

- Each rational number in the tree is irreducible.

- Each rational number appears in the tree only once.

- Every positive rational number appears in the tree (i.e. there are no rational numbers absent).

A more detailed discussion of the Stern-Brocot tree and proof of its properties is provided in [5], pp. 116-123.

5.9.2   The Correspondence Between The Stern-Brocot Tree And The Apparatus Of Continued Fractions

The Stern-Brocot tree, on examination, bears a clear resemblence to the apparatus of continued fractions. For example, in examining Figure 5.1 and following leftmost branches in the tree, the rational numbers 1/2, 1/3, 1/4, and 1/5 correspond respectively to the continued fractions [0;2], [0;3], [0;4], and [0;5]. Similarly, following the right branches down from 1/2 yields, in order, [0;1,2], [0;1,3], and [0;1,4]. Clearly, a relationship between the Stern-Brocot tree and the apparatus of continued fractions may exist.

Suspicions of a simple relationship may also arise by noting that the way in which the Stern-Brocot tree is constructed when only left branches or only right branches are pursued is of the same form as the rule for the formation of continued fraction convergents (Eqns. 5.19 and 5.20). For example, the nth successor to the right of 1/3 has the form

$$\frac{n+1}{2n+3}, \tag{5.93}$$

which is suspiciously similar to (5.19) and (5.20).

There is, in fact, an intimate relationship between the Stern-Brocot tree and the apparatus of continued fractions. We present this relationship as Lemma 5.35, below.

Lemma 5.35: Let $R^{z_0} L^{z_1} \ldots L^{z_{j-2}} R^{z_{j-1}} L^{z_j}$ or $R^{z_0} L^{z_1} \ldots R^{z_{j-2}} L^{z_{j-1}} R^{z_j}$ (depending on whether the final leg of the path is towards the left or towards the right, respectively) be the path in the Stern-Brocot tree from the fraction $1/1$ to the fraction a/b, where L^N denotes traveling downward and to the left in the tree N nodes, and R^N denotes traveling downward and to the right in the tree N nodes. Then the continued fraction representation of a/b is $[z_0; z_1, \ldots, z_{j-2}, z_{j-1}, z_j + 1]$.

Proof: The proof is inductive. First note that the constraints of the path require that $z_0 \geq 0$, and that $z_k \geq 1$, $k > 0$. In other words, only the first rightward leg of the path can have zero steps.

If the path is R^{z_0}, $z_0 = 0$, then the lemma predicts that the continued fraction representation will be $[z_0 + 1] = [1]$, which is the correct continued fraction representation of the fraction $1/1$. Note that the rational number $1/1$ has no ancestor in the tree.

If the path is R^{z_0}, $z_0 \neq 0$, then the lemma predicts that the continued fraction representation will be $[z_0 + 1]$, which is correct on inspection since the most rightward path in the Stern-Brocot tree traverses the non-negative integers. Note that the immediate ancestor of the fraction $[z_0 + 1]$ is $[z_0]$.

If the path is R^{z_0}, L^{z_1}, $z_0 = 0$, then the fraction a/b will be the weighted mediant of $1/1$ and $0/1$,

$$\frac{1}{z_1 + 1} = [0; z_1 + 1], \tag{5.94}$$

which argrees with the lemma. Note that the immediate ancestor ancestor of $[0; z_1 + 1]$ in the tree is $[0; z_1]$.

If the path is R^{z_0}, L^{z_1}, $z_0 \neq 0$, then the fraction a/b will be the weighted mediant of $(z_0 + 1)/1$ and $z_0/1$, i.e.

$$\frac{(z_0 + 1) + (z_0 z_1)}{(1) + (z_1)} = z_0 + \frac{1}{z_1 + 1} = [z_0; z_1 + 1], \tag{5.95}$$

which is consistent with the lemma. Note also that the immediate ancestor of the rational number specified by (5.95) is

$$\frac{(z_0 + 1) + z_0(z_1 - 1)}{(1) + (z_1 - 1)} = z_0 + \frac{1}{z_1} = [z_0; z_1]. \tag{5.96}$$

The cases with two or fewer path components have been proved above. It remains to prove all cases with three or more path components.

Let $s_k = p_k/q_k$ denote the kth-ordered convergent of the continued fraction $[z_0; z_1, \ldots, z_{j-1}, z_j]$ (note that the final partial quotient is *not* adjusted upwards by one). For $k \geq 2$, we can establish a relationship between $[z_0; z_1, \ldots, z_{k-1}, z_k]$ and $[z_0; z_1, \ldots, z_{k-1}, z_k + 1]$ as follows:

$$[z_0; z_1, z_2, \ldots, z_{k-2}, z_{k-1}, z_k + 1] = \frac{(z_k + 1)p_{k-1} + p_{k-2}}{(z_k + 1)q_{k-1} + q_{k-2}} = \frac{p_k + p_{k-1}}{q_k + q_{k-1}}. \quad (5.97)$$

If we agree for convenience, as was mentioned in Section 5.5, that we will define $s_{-1} = p_{-1}/q_{-1} = 1/0$, then (5.97) holds for $k \geq 1$.

(Inductive Step): Assume that the lemma holds up through $k-1$. For a path in the Stern-Brocot tree $R^{z_0} L^{z_1} \ldots L^{z_{k-2}} R^{z_{k-1}} L^{z_k}$ or $R^{z_0} L^{z_1} \ldots R^{z_{k-2}} L^{z_{k-1}} R^{z_k}$, $k \geq 2$, the "reversal" fraction above (i.e. the fraction where the path changes directions) is

$$[z_0; \ldots, z_{k-2}, z_{k-1} + 1] = \frac{p_{k-1} + p_{k-2}}{q_{k-1} + q_{k-2}} \quad (5.98)$$

(this is established by the lemma on the path through $k-1$ and by Eqn. 5.97).

The immediate ancestor of the fraction specified in (5.98) is

$$[z_0; \ldots, z_{k-2}, z_{k-1}] = \frac{z_{k-1}p_{k-2} + p_{k-3}}{z_{k-1}q_{k-2} + q_{k-3}}, \quad (5.99)$$

as was shown to hold in (5.96) and in the inductive step.

The rational number corresponding to the path $R^{z_0} L^{z_1} \ldots L^{z_{k-2}} R^{z_{k-1}} L^{z_k}$ or $R^{z_0} L^{z_1} \ldots R^{z_{k-2}} L^{z_{k-1}} R^{z_k}$ is a weighted mediant of (5.98) and (5.99):

$$\frac{z_k(z_{k-1}p_{k-2} + p_{k-3}) + p_{k-1} + p_{k-2}}{z_k(z_{k-1}q_{k-2} + q_{k-3}) + q_{k-1} + q_{k-2}} = \frac{(z_k + 1)p_{k-1} + p_{k-2}}{(z_k + 1)q_{k-1} + q_{k-2}} \quad (5.100)$$

$$= \frac{p_k + p_{k-1}}{q_k + q_{k-1}} \quad (5.101)$$

$$= [z_0; z_1, \ldots, z_{k-1}, z_k + 1], \quad (5.102)$$

which establishes the main result of the lemma in the inductive step. Note also that the immediate ancestor of the fraction specified in (5.100) is $[z_0; \ldots, z_{k-1}, z_k]$ (which is necessary for the inductive step). This proves the lemma. ∎

Remarks: This lemma provides a straightforward method to go from a fraction's position in the Stern-Brocot tree to its continued fraction representation, or vice-versa.

To go from a fraction's position in the Stern-Brocot tree to its continued fraction representation:

- Starting with moves downward and to the right from the fraction $1/1$ (z_0), observe the length of the alternating rightward and leftward node traversals required to reach the desired fraction.

- Adjust the final length upward by one.

- These lengths in order are then the successive partial quotients of the fraction.

To go from the continued fraction representation of a fraction to its position in the Stern-Brocot tree:

- Reduce the final partial quotient by one.

- Use the partial quotients, in order, in an alternating fashion, to go rightward and downward and leftward and downward in the Stern-Brocot tree. The fraction reached on the final leg will be the fraction of interest.

It is clear from the lemma above that the Stern-Brocot tree and the apparatus of continued fractions are intimately related. Specifically, the Stern-Brocot tree provides a model for the formation and arrangement of rational numbers, but the apparatus of continued fractions provides a much more efficient way to navigate the Stern-Brocot tree and to find best rational approximations.

5.9.3 Using The Stern-Brocot Tree To Find Best Rational Approximations

It is clear from Example 5.34 that the Stern-Brocot tree can be used to find best rational approximations in the Farey series of any order, simply by forming mediants until the number of interest is enclosed. It is also clear that an algorithm of repeatedly forming mediants in order to find a best rational approximation in $F_{k_{MAX}, \overline{h_{MAX}}}$ can be devised.

However, the sole drawback of such a procedure is that building the Stern-Brocot tree from the top in order to find neighbors in F_N is an $O(N)$ procedure, which renders it unsuitable for use with large N. For this reason, the continued fraction algorithms presented earlier in the chapter, which are $O(log\ N)$ (due to the guaranteed minimum exponential rate of increase of convergent denominators, Theorem 5.12), are the only practical algorithms.

5.10 Practical Techniques

Although this chapter has presented rather theoretical results and techniques from number theory, our emphasis is on practical applications (which is why

we've concentrated on finding Farey neighbors of *rational* numbers). Practicing engineers would be more likely to use the digits from a calculator as the starting point to obtain a rational approximation than to use a symbolic irrational constant, such as π.

In this section, we present *practical* techniques—those most likely to be used in practice by microcontroller software developers.

5.10.1   Practical Aspects Of Beginning With A Rational Approximation

In practical applications, one often begins with a rational approximation to a irrational number. For example, one might use 3.14159265359 (from a calculator) as the value of π for the application of Algorithm 5.1. This naturally raises the question of how accurate the rational approximation used as a starting point must be to avoid identifying the wrong rational numbers as Farey neighbors. We illustrate the possibility of identifying the wrong rational number with an example.

Example 5.36:   Find the members of F_{255} which enclose π, using 3.1416 as the value of π.

[Erroneous] Solution And Remarks: It can be shown using the methods presented earlier that the members of F_{255} which enclose 3.1416 are $355/113$ and $732/333$, i.e.

$$\frac{355}{113} < 3.1416 < \frac{732}{333}. \tag{5.103}$$

However, in fact, $688/219$ and $355/113$ are the actual enclosing neighbors of π in F_{255}:

$$\frac{688}{219} < \pi < \frac{355}{113} < 3.1416 < \frac{732}{333}. \tag{5.104}$$

Thus by using an imprecise approximation of π, we have incorrectly identified the neighbors to π in F_{255}.

How do we avoid incorrectly identifying the rational numbers which enclose an irrational number when a rational approximation of the irrational number is used as the starting point for the selection algorithm? There are three practical approaches to the problem.

Observe that when we know the first several decimal digits of an irrational number, the actual value of the irrational number is confined to an interval. For example, if a calculator displays "3.1416" as the value of π, we might safely assume that $3.14155 \le \pi \le 3.14165$, if the digits that the calculator displays were obtained by rounding.

As a first approach to dealing with a rational approximation to an irrational number, we might simply determine the Farey neighbors of both endpoints of the interval of uncertainty. For example, if 314,155/100,000 and 314,165/100,000 have the same Farey neighbors in a Farey series of interest (which we can easily determine using the algorithms presented earlier in this chapter), we could correctly deduce that these Farey neighbors are the Farey neighbors of π. On the other hand, if 314,155/100,000 and 314,165/100,000 have different enclosing Farey neighbors, then there are Farey terms in the interval [3.14155, 3.14165], and more information about π would be required to determine its true Farey neighbors.

A second approach to this same problem would be to devise an algorithm to process the endpoints of the interval of uncertainty simultaneously and note when their partial quotients diverge.

A third approach, which is perhaps the most direct, is to apply Algorithm 4.14 to determine the rational number with the minimum denominator in the interval of uncertainty. We would thus know that we have enough information to determine uniquely the enclosing rational numbers in any Farey series up to one less than this minimum denominator.

Example 5.37: Assume that 3.142 is the only value for π available. What is the maximum order of the Farey series where the enclosing rational numbers to π can be unambiguously determined?

Solution: Assume that the constant "3.142" was obtained by rounding of digits, rather than by truncation of digits: thus $3.1415 \leq \pi \leq 3.1425$. Applying Algorithm 4.14 yields 333/106 as the rational number with the smallest denominator in the interval [3.1415, 3.1425]. Thus, no rational number with a smaller denominator exists in the interval, and the enclosing rational numbers of π in the Farey series of up to order 105 can be determined with the limited information available.

5.10.2 Obtaining Irrational Numbers To High Precision

It may happen in practice that one desires more information about an irrational number than can be easily obtained. As a practical example, a typical scientific calculator treats π as 3.14159265359, implying that $3.141592653585 \leq \pi \leq 3.141592653595$. Application of Algorithm 4.14 to this interval yields 1,146,408/364,913 as the rational number with the smallest denominator in this interval: implying that we cannot determine the enclosing rational approximations to π even in $F_{2^{32}-1}$ using the information most readily available. How do we obtain more digits of π? And even if we can obtain more digits of π, how do we manipulate rational numbers with such large integer components? (We discuss the problem of obtaining more digits below, but discuss manipulation in Section 5.10.3.)

There are two approaches to determining π or other numbers to high precision.

1. Locate information about the number on the Web or in a reference book. (Note that decimal digits of the number, partial quotients of the number, or convergents of the number can all be used; and it is typical for all of these to be somewhere on the Web. However, convergents are the most useful form for obtaining best rational approximations.)

2. Use commercial symbolic manipulation software (*Mathematica* [75], for example) to obtain the number of interest to arbitrary precison.[30]

5.10.3   Manipulating High-Precision Rational Numbers

Assuming that one is able to determine a [rational or irrational] number of interest to high-precision (either a large number of decimal digits or a rational number with large integer components), how does one manipulate rational numbers with large integer components? In this section, we list software alternatives.

The first alternative we should mention is *The Iju Tool Set*, distributed with this book. Starting with version 1.05, this tool set contains a subset of the GNU MP Library [73], and will manipulate large integers and rational numbers with large integer components. To provide more flexibility for the user, this tool set is embedded as extensions to the Tcl scripting language; so that any of the functionality provided can be used either interactively or from within a Tcl script.

Figure 5.2 shows a screen snapshot[31] of *IjuConsole* (the *Wish*-like Tcl interpreter from *The Iju Tool Set*) being used interactively to provide the answers to several questions involving large integers and rational numbers with large integer components. The first command shown,

```
arbint intmul 218643832695416243621 13254215384521848,
```

illustrates integer multiplication. The second command shown,

```
arbint intfac 55,
```

illustrates calculating the factorial of 55. The third command shown,

```
arbint cfbrapab [arbint const pi 500] 65535 65535,
```

illustrates calculating the best rational approximation to π with numerator and denominator not exceeding 65,535, and using the first 500 digits of π as the value of π. The fourth command shown,

---

[30] *Mathematica* [75] is quite expensive (version 4.1 for Windows, as of March 2001, is \$1,495), and something that a microcontroller software developer would very rarely use, so we assume that most microcontroller software developers would search for a less expensive solution.

[31] By the way, *IjuConsole* will handle *much* larger integers, but small examples were used so that all of the information would fit in a single screen snapshot.

Figure 5.2: Large Integer Arithmetic And Best Rational Approximations Using *IjuConsole* From *The Iju Tool Set*

```
arbint cfbrapab 1.609344 1023 1023,
```

illustrates finding the best rational approximation to the conversion factor from MPH to KPH with numerator and denominator not exceeding 1,023. The last command shown,

```
arbint rnadd 78/2976 342/2763,
```

illustrates the addition of two rational numbers.

Many other potential solutions for dealing with large integers have been submitted by newsgroup posters, and are listed below. (Please note that these alternatives haven't actually been tried, and we can't say whether they are viable.)

1. *Mathematica* [75] (by Wolfram Research) will easily operate on large integers and rational numbers with large integer components (see Footnote 30). (Suggested by Paul Lutus [47] and Don Taylor [55].)

2. The *GNU Multiple Precision Arithmetic Library (GMP)* [73]. This library, which is free and on the Web, can be linked into 'C' and 'C++' programs, and allows fast integer calculations of any size that do not exceed the memory available in the computer. This library could be used to quickly construct a program to process rational numbers with very large integer components.

3. *UBASIC* [77] (by Yuji Kida [27]) is an extended-precision version of the BASIC language which will handle integers up to 2,600 digits and exact rational arithmetic. (Suggested by Richard Schorn [52].)

4. *Derive 5* [72] (by Texas Instruments). The exact capabilities of this software are not known, but the Web page indicates it can perform exact rational arithmetic. (Suggested by Richard Schorn [52] and Don Taylor [55].)

5. *Maple* [74] (by Waterloo Maple, Inc.). The exact capabilities of this software are not known. (Suggested by Paul Lutus [47] and Don Taylor [55].)

6. *MuPAD* [76] (from the University Of Paderborn, Germany). The capabilities of this software are not known. (Suggested by Richard Schorn [52].)

In addition to the large integer resources above, a much longer list of resources is maintained at the URL

`http://www.csc.fi/math_topics/Mail/FAQ/msg00015.html`,

and is reproduced below. Because this URL was apparently last updated in 1994, it is not known which of the resources listed are still available.

```
--------------------------------------------------------------------------
Subject: List of Arbitrary Precision C packages
From: mrr@scss3.cl.msu.edu (Mark Riordan)
Date: 27 Jan 1994 16:06:01 GMT
Newsgroups: sci.math
--------------------------------------------------------------------------
This is the file BIGNUMS.TXT from ripem.msu.edu, last updated January 1994.

In response to Email requests, I have assembled this list of
large-integer arithmetic packages of which I have heard.
Most of these are C function libraries, available in source form.
A few also deal with floating point numbers.

For your convenience, I have placed copies of
some of these on ripem.msu.edu (35.8.1.178).  They are
available for anonymous FTP in the directory "pub/bignum".
However, what I have may not be the most current version in all cases.

Here they are, in no particular order:

mp
    Multiple Precision package that comes with some Unixes

    Multiple precision package accessed via -lmp flag on your
    compiler.  Provides +, -, *, /, gcd, exponentiation,
    sqrt.  Comes with SunOS, NeXT Mach, BBN Mach 1000,
    and probably a few others.  See "man 3 mp".
    Object code only, of course.

PARI
    Henri Cohen, et al., Universite Bordeaux I, Paris, FRANCE

    Multiple precision desk calculator and library routines.
    Contains optimized assembly code for Motorola 68020,
    semi-optimized code for SPARC, and apparently rather slow
    generic C version.  Does both integers and reals.
```

```
     Does vectors and matrices as well as scalars.
     Contains a number of advanced functions, some of which I've
     never heard of.  ("Weber's function"?)
     Has a factorization function, primality test, & other related stuff.
     Plenty of TEX documentation.
     Public domain, but you can't distribute modified versions.
     Available via anonymous FTP from ftp.inria.fr:lang/ and
     math.ucla.edu.  The ucla site has Mac, MSDOS, OS/2, and
     NeXT-specific versions there in addition to:
     Filename:  pari-1.37.tar.Z  (There are now more recent versions)

Arithmetic in Global Fields  (Arith)
     Kevin R. Coombes, David R. Grant

     Package of routines for arbitrary precision integers or
     polynomials over finite fields.  Includes basic +, -, *, /
     and a few others like gcd.  Source code in C.
     Distributed under the terms of the GNU public license.
     Includes man pages and TEX documentation.
     Filename:  arith.tar.Z

Arbitrary Precision Math Library
     Lloyd Zusman   Los Gatos, CA

     C package which supports basic +, -, *, /.  Provides for radix
     points (i.e., non-integers).  Not as polished as the others here.
     Posted to comp.sources.misc in October 1988.
     Filename:  apml.tar.Z

BigNum
     J. Vuillemin, INRIA, FRANCE, and others.
     Distributed by Digital Equipment Paris Research Lab (DECPRL)

     A "portable and efficient arbitrary-precision integer" package.
     C code, with generic C "kernel", plus assembly "kernels" for
     MC680x0, Intel i960, MIPS, NS32032, Pyramid, and of course VAX.
     This is probably one of the better-known packages of this type.
     Implements +, -, *, /, mod, plus logical operations OR, AND, XOR.
     Both signed and unsigned arithmetic available.
     Available via email from librarian@decprl.dec.com.
     You will receive 5 shell archives.  Give your postal address
     and you will also receive printed documentation from France.
     Package includes TEX documentation.
     Publicly available for non-commercial use.
     I removed this from my archive when I heard a rumor that PRL
     doesn't like others to distribute it.  However, BIGNUM *is*
     distributed as part of ecpp (see below).

Lenstra's LIP package
     Arjen Lenstra   Bellcore

     Portable unsigned integer package written entirely in C.
     Includes +, -, *, /, exponentiation, mod, primality testing,
     sqrt, random number generator, and a few others.
     An earlier version of this package is the only of these packages
     I have actually used.  It works well and is very portable.
     I haven't done any benchmarks against the others, but the code
     looks clever & Lenstra is an accomplished number theorist.

     LIP replaces lenstra-3.1.c.  The package now includes encrypted
     source code; to obtain the decryption key, you must send a
     signed license agreement to Bellcore.  See the documentation.
     Filename:  lenstra-LIP-package.tar  This is a collection of
     all the files in flash.bellcore.com:/pub/lenstra

bmp  (Brent's Multiple Precision?)
     R. P. Brent
```

```
            1981 vintage FORTRAN code to do extended precision floating &
            fixed point arithmetic.  Includes most of the mathematical
            functions you'd find in a FORTRAN run-time library.
            This code is an ACM algorithm, number 524.
            To obtain, send a mail message to  netlib@ornl.gov
            containing the line "send mp.f from bmp" or better yet, perhaps
            just start with "help".

    SPX
        Kannan Alagappan & Joseph Tardo, DEC

        This is a huge prototype public key authentication system
        based on RSA.  I mention it here because those who have heard
        of SPX have probably correctly guessed that it contains a large
        integer package and I want to inform you that the large integer
        package it contains is indeed DEC's BigNum from France.
        You can get a beta test copy of SPX from crl.dec.com (192.58.206.2).
        Use it only for testing, as it "may" expire on a certain date.
        (I don't know whether this has expired yet.)

    amp  (Antti's Multiple Precision?)
        Antti Louko   alo@kampi.hut.fi

        Multiple precision integer package in C.  Includes +, -, *, /, %,
        pow, mod, 1/x mod y, random, sqrt, gcd.  Available for non-commercial
        use.  The package includes "share-secret", a public key system based
        on the Diffie-Hellman algorithm.
        This is normally part of the well-known "des-dist.tar.Z",
        but I have removed the DES part to avoid having to deal with
        cryptographic export laws, and have named the result:
        Filename:  amp.tar.Z

    gennum
        Per Bothner   U of Wisconsin-Madison

        C++ routines and classes to do generic arithmetic, both
        integer and rational.  Part of the "Q" programming system.
        Distributed under the terms of the GNU public license.
        Obtained from cygnus.com.
        Filename: gennum.tar.Z

    MIRACL
        (Shamus Software, Dublin, Ireland)

        Integer and fractional multiple precision package.
        MIRACL is a portable C library. Full C/C++ source code included
        (In-line assembly support for 80x86). Number theoretic primitives
        needed to support PK Cryptography are supplied.
        C++ classes for Multiprecision Integers, Modular arithmetic, and
        Chinese Remainder Thereom. Implementation in C/C++ of all modern
        methods of Integer Factorisation, viz Brent-pollard, p-1, p+1,
        Elliptic Curve, MPQS.  Includes TEX manual and some DOS .EXEs.
        Not public domain, but free for academic and non-commercial use.
        Obtained from ftp.compapp.dcu.ie.
        Filename:  /pub/crypt/other/miracl-3.23.zip and miracl.tar.Z (older)

    precision
        Dave Barrett  barrettd@tigger.colorado.edu

        Multiple precision integer package in C with +,-,*,/, sqrt, rand,
        mod, pow, log.  Simple vector support.  Does dynamic allocation of memory.
        Free as long as you don't sell it or any program that uses it.
        Filename:  precision.tar.Z

    UBASIC
        Prof. Yuji Kida, Rikkyo University, Nishi-Ikebukuro 3, Tokyo 171, Japan
        kida@rkmath.rikkyo.ac.jp
```

Multiple-precision version of the BASIC programming language,
for MS-DOS. Includes floating point. Said (by Keith Briggs)
to be pretty fast. Object only, I think. ervin@morekypr.bitnet
says: "This is the best package that I know of for
fast arithmetic. Has a version optimized for 386 machines. Includes
routines to do MPQS, the fastest currently known general factoring
algorithm. An additional file is at both sites to allow MPQS to use
hard drives so that it can factor up to 80 digits. Many number
theoretical functions are included in UBASIC. It allows over 2500
digits of precision."
Available via anonymous FTP from shape.mps.ohio-state.edu,
or simtel20.army.mil, or wuarchive.wustl.edu.

calc_v22
    Unknown

    MS-DOS C-like language that allows "infinite" precision.
    Nice intrinsic functions. ervin@morekypr.bitnet reports problems
    when changing precision on the fly.
    See simtel20 or wuarchive.

briggs_arith
    Keith Briggs (kbriggs@maths.adelaide.edu.au)

    Turbo Pascal 5 source for routines that do multiple-precision
    +, -, *, /, sqrt, gcd, factoring, rand for integers; also includes
    +, -, *, / and rand for rational numbers.
    Filename: briggs_arith.pas

Institute fur Experimentelle Mathematik
    Dr Gerhard Schneider (?)

    Fast C multiple-precision subroutine library.
    I don't know anything about it; sl25@ely.cl.cam.ac.uk says
    to contact MAT420@DE0HRZ1A.BITNET for more info.
    Postal Address:
    Institute fur Experimentelle Mathematik
    EllernStr 29
    D4300 Essen-12   GERMANY

LongInt
    Markus Mueller (mueller@komsys.tik.ethz.ch)

    "Multi precision arithmetic written in MODULA-2, with the most time critical
    parts written in Assembler. Includes basic arithmetics (+, -, *, /, %) as
    well as arithmetics MODULO a number. An additional module provides a
    collection of procedures for primality testing, gcd, multiplicative
    inverse and more. The package is part of a Privacy Enhanced Mail (PEM)
    package which includes a PEM mailer, RSA key generator and Certificate
    generation tools."

    Source is in Modula-2, C, and assembler for Sun 3. LongInt has
    also been ported to MS-DOS under Logitech Modula-2 and Turbo
    Assembler. Availability: free for university use (research and
    education); otherwise, a source license is required. To obtain,
    write or email to:

        Markus Mueller
        Bertastrasse 7
        CH-8953 Dietikon
        Switzerland
        email: mueller@komsys.tik.ethz.ch

bignum-1.2
    Henrik.Johansson@Nexus.Comm.SE

    Bignum package written in portable C. Will in the future
    conform to the Common Lisp functions that handles integers.

```
        Currently includes +, -, *, /, exponentiation, "exptmod",
        comparison, random numbers, and gcd.
        Filename: bignum-1.2
```

ACM algorithm 567
    D.W. LOZIER and J.M. SMITH

```
    FORTRAN subroutines to do extended-precision floating point
    and normalized Legendre polynomials.
    ACM TRANSACTIONS ON MATHEMATICAL SOFTWARE 7,1 (MARCH, 1981)
    Obtained from research.att.com:netlib/toms/567.Z
    Filename: acm-algorithm-567-floating-point.fortran.Z
```

range
    O. Aberth and M. J. Schaefer

```
    C++ package to do extended-precision floating point arithmetic
    with programmer-defined precision.  Uses decimal representations
    internally.  Contains basic +, -, *, /, relational operators,
    ++, and a few functions like sin, cos, sqrt, log.  Documentation
    a trifle confusing.
    Obtained from math.tamu.edu:pub/range/range.tar.Z
    Filename: range.tar.Z
```

bsint
    Author unknown.

```
    Pre-alpha release of C++ big integer package.
    Implements basic math operators, exponentiation, and modular
    exponentiation.  Very skimpy documentation.
    See milton.u.washington.edu:/pub/user-supported/tzs/bsint.tar.Z
```

GNU Multiple Precision (GMP)
    GNU (Free Software Foundation) multiple precision package.
    I haven't looked at it yet. This is current as of April 1992,
    but there may be a more recent version by the time you read
    this. This package is very widely available on FTP sites.
    Filename: gmp-1.3.2.tar.Z

libg++ - GNU's C++ class library
    Free Software Foundation

```
    Includes Integer and Rational classes.  Integer provides
    the usual C++ operators, plus exponentiation, gcd, lcm.
    Limited functionality, but documentation is better than most.
    Look for libg++-2.4.tar.gz on an FTP server near you.
```

Elliptic Curve Primality Proving
    Francois Morain, France.

```
    Large package to prove the primality of any prime.
    Includes Inria's BIGNUM package.
    Obtained from ftp.inria.fr (128.93.1.26).
    Filename: ecpp.V3.4.1.tar.Z
```

PGP (Pretty Good Privacy)
    Philip Zimmermann   prz@sage.cgd.ucar.EDU

```
    Crypto package that includes bignum routines in C.
    Assembly implementations available for several processors;
    said to be quite fast for numbers around 1000 bits in size.
    The crypto package violates RSA patents, but the bignum routines
    can be used without fear of legal repercussions.
```

Bell's Arbitrary Precision Calculator
    David I. Bell, Australia (dbell@pdact.pd.necisa.oz.au)

```
    Arbitrary-precision calculator with good online help, C-like
```

```
        language, many builtin functions, support for integers,
        rational numbers (they work like floating point), complex numbers,
        matrices, strings, lists, files, "objects".  Includes
        gcd, primality testing, even trig functions.  Recommended.
        (Large package, though.)  Obtained from comp.sources.unix.
        Filename: calc-1.24.7.tar.Z

Calc for GNU Emacs
    Dave Gillespie  (daveg@synaptics.com)

    Advanced calculator written in Emacs Lisp.  Includes arbitrary
    precision integers and floating point, bitwise operations,
    log and trig functions, financial functions, number theoretic
    functions including prime factorization, symbolic calculus,
    and an interface to GNUPLOT.
    Filename:  calc-2.02a.tar.Z

MPFUN: A Multiple Precision Floating Point Computation Package
    David H. Bailey   (dbailey@nas.nasa.gov)

    Package of Fortran subroutines to perform multiprecision
    floating point arithmetic.  Also includes a program that
    can automatically convert ordinary Fortran-77 code into code
    that calls the MPFUN routines.
    Keith Briggs says: "It's a masterpiece, and the state of the art
    as far as Fortran goes."
    Documentation in TeX format.  Unrestricted distribution
    allowed at no cost.
    Filenames: mpfun_fortran.tar.Z & mpfun_tex_papers.tar.Z

MPQS
    Mark S. Manasse  (msm@src.dec.com)  and Arjen Lenstra

    C program to factor numbers on a distributed network of
    heterogeneous machines.  June 1993 version.
    Filename:  mpqs-distributed-factoring.shar

GNU bc
    Author: Philip A. Nelson (phil@cs.wwu.edu)
    GNU bc is an interactive algebraic language with arbitrary precision.
    GNU bc is almost the same as bc & dc in some Unixes.
    Filename: bc-1.02.tar.z  (for example, in GNU prep.ai.mit.edu:pub/gnu/)

bc & dc
  bc is an interactive processor for an arbitrary precision arithmetic
  language or just compiler/preprocessor for dc calculator with arbitrary
  precision; they comes with some Unixes.

Built-in support in other languages
    Various

    Multiple precision arithmetic is available in a number of
    programming languages, such as Lisp and ABC (cf. mcsun.eu.net).
    Version 8 of the programming language Icon (Griswold's successor
    to SNOBOL4 available from cs.arizona.edu) has large integers.
    Perl (by Larry Wall, available from devvax.jpl.nasa.gov)
    includes source, in Perl, for such a package, but it's probably
    not suitable for serious use.
    For some of these, source code may be available.  This list is
    long enough, so I'm not going to pursue it aggressively.

Thanks to Keith Briggs and several others who contributed to this list.

See also other sites, such as nic.funet.fi:pub/sci/math/multiplePrecision/.

Mark Riordan   mrr@ripem.msu.edu
------------------------------------------------------------------------------
```

5.11   Authors And Acknowledgements

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5.12   Exercises

5.12.1   Algorithms

[**5.1**] Develop an algorithm to convert a continued fraction $[a_0; a_1, \ldots, a_n]$ to a rational number a/b "from the right" (starting with a_n), and prove that the a/b generated will be irreducible. (Hint: the ordinary algorithm often applied by hand—working "from the bottom up" as shown in Example 5.4 will always generate a coprime a/b.)

5.12.2   Calculation Of Best Rational Approximations

[**5.2**] Assuming 1.609344^{32} as the exact conversion factor from miles to kilometers, find the best rational approximation to this conversion factor with a maximum numerator of 255 and a maximum denominator of 255.

[**5.3**] Assuming 1.609344 (see Footnote 32) as the exact conversion factor from miles to kilometers, find the best rational approximation to this conversion factor with a maximum numerator of 65,535 and a maximum denominator of 65,535.

5.12.3   Continued Fraction Representation Of Irrational Numbers

[**5.4**] Show that the continued fraction representation of the golden ratio ($\sqrt{5}/2+ 1/2$) is $[1; \overline{1}]$.

---

[32]This conversion factor was obtained from [78] and is assumed to be the most accurate conversion factor available.